Reconstruction of a function from Hermite-Birkhoff data

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Abstract

Birkhoff (or lacunary) interpolation is an extension of polynomial interpolation that appears when observation gives irregular information about a function and its derivatives. A Birkhoff interpolation problem is not always solvable even in the appropriate polynomial or rational space. In this paper we split up the initial problem in subproblems having a unique polynomial solution and use multinode rational basis functions in order to obtain a global interpolant.

Keywords: Birkhoff interpolation, Rational approximation, Remainder term, Order of approximation.

1. Birkhoff interpolation

In 1906 G. D. Birkhoff\textsuperscript{3} studied the problem related to lacunary interpolation, that is interpolation which appears whenever observation gives irregular information about a function and its derivatives. Few years later, in 1931, G. Polya\textsuperscript{2} gave a notable contribution by introducing certain algebraic inequalities that must be satisfied by the interpolation scheme to be regular, i.e. solvable for each choice of pairwise distinct nodes and associated interpolation data. These papers received little attention until I. J. Schoenberg revived interest on the subject in 1966\textsuperscript{3}, when he provided a generalization of the Polya’s theorem which gives a necessary condition to the existence of the solution. Lacunary or Birkhoff interpolation, in polynomial space, radically differs on Lagrange or Hermite interpolation in both its problems and its methods\textsuperscript{4}. 

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More precisely, let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of pairwise distinct real numbers for which we assume that $x_1 < x_2 < \cdots < x_n$. In the problem of interpolation of given data $f_{i,j} = f^{(j)}(x_i), i = 1, \ldots, n, \; j \in J_i \subset \mathbb{N}$, by a polynomial $p$ of appropriate degree,

$$p^{(j)}(x_i) = f_{i,j}$$

we mainly distinguish between Hermite interpolation and Birkhoff interpolation. We have an Hermite interpolation problem if, for each $i$, the indices $j$ in the set $J_i$ form an unbroken sequence, i.e. $J_i = \{0, 1, \ldots, j_i\}$, a Birkhoff interpolation problem otherwise. In contrast to Hermite interpolation, a Birkhoff interpolation problem does not always have a unique solution or, even worse, does not have a solution. It is, however, convenient to consider Hermite interpolation to be a special case of lacunary interpolation and to deal with Hermite-Birkhoff interpolation.

For instance, there is no quadratic polynomial $p(x) = ax^2 + bx + c$ such that

$$p(-1) = p(1) = 0, \quad p'(0) = 1. \quad (1)$$

In this case we can try to enlarge the space of possible solutions by considering rational functions

$$\left(\frac{p}{q}\right)^{(j)}(x_i) = f_{i,j},$$

instead of polynomials, hoping that the problem is solvable in the larger space. In [5] univariate Birkhoff rational interpolation problem is investigated. Firstly, the Birkhoff rational interpolation problem is converted into a polynomial system solving problem. Then the polynomial system is solved by means of Groebner basis and thus the solution of the Birkhoff rational interpolation is obtained. However, by easy calculations, we can see that the problem

$$\left(\frac{p}{q}\right)(-1) = \left(\frac{p}{q}\right)(1) = 0, \quad \left(\frac{p}{q}\right)'(0) = 1 \quad (2)$$

has not a solution in the space of rational functions of the form

$$\left(\frac{p}{q}\right)(x) = \frac{ax + b}{x + c},$$
neither in the space of rational functions of the form
\[
\left(\frac{p}{q}\right)(x) = \frac{a}{x^2 + bx + c}
\]
which are appropriate spaces to consider if we are looking for a unique solution of (2). The interest in this kind of interpolation lies in the fact that, in recent years, many scholars applied the Birkhoff interpolation in numerically solving boundary value problems or initial-value problems and rational functions sometimes are superior to polynomials with the same interpolation data because they can achieve more accurate approximations with the same amount of computation (see [6] and the references therein).

In this paper we propose to split up the unsolvable problems in two or more solvable subproblems, as shown in Figure 1 for the particular case in (1), whose solutions can be blended together. Here we consider the case of multinode basis functions as blending functions. To this goal we consider a covering \( F = \{F_1, F_2, \ldots, F_m\} \) of \( X \) by subsets \( F_k \subset X \) such that, for each \( k = 1, \ldots, m \), the corresponding Hermite-Birkhoff interpolation subproblem \( p^{(j)}(x_i) = f_{i,j}, x_i \in F_k, j \in J \) has a unique solution and we associate to each \( F_k, k = 1, \ldots, m \), a multinode basis function (see Section 2). The latter are then used in combination with the local Hermite-Birkhoff polynomials that interpolate the data associated to \( F_k \) (see Section 3 and 4), and the approximation order of the combination is studied (see Section 6.1). Finally, we provide numerical experiments which confirm the theoretical results on the approximation order and show a good accuracy of approximation (see Section 6).

2. Multinode basis functions

Let us consider a covering \( F = \{F_1, F_2, \ldots, F_m\} \) of \( X \) by its not empty subsets \( F_k \subset X \), that is
\[
\bigcup_{k=1}^{m} F_k = X, \quad F_k \neq \emptyset, \quad \text{for each } k = 1, \ldots, m. \quad (3)
\]
The idea

domain decomposition

\begin{align*}
\ast & \ast \ast \ast \\
\ast & \ast \ast \ast
\end{align*}

no solution

\begin{align*}
\ast & \ast \ast \\
\ast & \ast \ast
\end{align*}

unique solution

Figure 1: The unsolvable Hermite-Birkhoff problem (on the left) is splitted up in two solvable subproblems (on the right). We denote by a ball the available data and by a cross the not-available data. The first line relates to $f(x_i)$, the second one to $f'(x_i)$.

The multinode basis functions with respect to the covering $\mathcal{F}$ are defined by

$$B_{\mu,k}(x) = \frac{\prod_{x_i \in F_k} |x - x_i|^{-\mu}}{\sum_{l=1}^{m} \prod_{x_i \in F_l} |x - x_i|^{-\mu}}, \quad k = 1, \ldots, m,$$

where $\mu > 0$ is a parameter that determines the differentiability class of the basis functions and controls the range of influence of the data values, in a sense that we specify later in Section 6 (see Figure 2). The multinode basis functions (4) are non-negative and form a partition of unity, that is

$$\sum_{k=1}^{m} B_{\mu,k}(x) = 1,$$

but instead of being cardinal they satisfy the following properties.

**Proposition 1.** The multinode basis functions in (4) vanish at all nodes $x_j$ that are not in $F_k$,

$$B_{\mu,k}(x_j) = 0, \quad \mu > 0,$$

for any $k = 1, \ldots, m$ and $x_j \notin F_k$, and

$$\sum_{k \in K_i} B_{\mu,k}(x_i) = 1, \quad \mu > 0,$$

where

$$K_i = \{l \in \{1, \ldots, m\} : x_i \in F_l\} \neq \emptyset,$$
is the set of indices of all subsets of $\mathcal{F}$ that contain $x_i$.

**Proof.** If we multiply both the numerator and the denominator of (4) with $|x - x_j|^\mu$, then

$$B_{\mu,k} (x) = \frac{C_k (x)}{\sum_{l=1}^{m} C_l (x)},$$

where

$$C_l (x) = |x - x_j|^\mu \prod_{x_i \in F_l} \frac{1}{|x - x_i|^\mu}, \quad l = 1, \ldots, m.$$  

Then, $C_l (x_j) = 0$ if and only if $l \notin K_j$, and (6) follows because $k \notin K_j$. Equality (7) follows by (6) and the partition of unity property (5).

**Proposition 2.** For $\mu > 0$ even integer the multinode basis functions (4) are rational and have no real poles, otherwise their class of differentiability is $\mu - 1$ for $\mu$ odd integer and $[\mu]$, the largest integer not greater than $\mu$, in all remaining cases. Moreover, all derivatives of order $\ell > 0$ vanish at all nodes $x_j$ that are not in $F_k$,

$$B^{(\ell)}_{\mu,k} (x_j) = 0,$$  

for any $k = 1, \ldots, m$ and $x_j \notin F_k$ and

$$\sum_{k \in K_i} B^{(\ell)}_{\mu,k} (x_i) = 0, \quad \mu > 1.$$  

**Proof.** If $\mu > 1$, then $C_l (x)$ is differentiable at $x_j$, and (9) follows because

$$B'_{\mu,k} (x) = \frac{C'_k (x) \sum_{l=1}^{m} C_l (x) + C_k (x) \sum_{k=1}^{m} C'_l (x)}{\left( \sum_{l=1}^{m} C_l (x) \right)^2}$$

and $C'_k (x_j) = 0$ for $x_j \notin F_k$. The procedure can be iterated until $C_l^{(\ell)} (x)$ exists at $x_j$. Equation (10) follows by differentiating both sides of (5) and by using equation (9).

**Proposition 3.** The multinode basis functions (4) may be written in the following form

$$B_{\mu,k} (x) = \frac{\prod_{x_i \notin F_k} |x - x_i|^\mu}{\sum_{l=1}^{m} \prod_{x_i \notin F_l} |x - x_i|^\mu}. \quad (11)$$
Figure 2: Multinode basis functions $B_{2,j}(x)$ and $B_{4,j}(x)$ for the subsets $F_k$ displayed with black balls.
Proof. Expression (11) is obtainable from (4) by elementary calculations.■

3. Multinode local Hermite-Birkhoff interpolation

Let us consider the Hermite-Birkhoff interpolation problem
\[ p^{(j)} [f] (x_i) = f^{(j)} (x_i), \quad i = 1, \ldots, n, \ j \in \mathcal{J}_i, \]  
(12)
and let us assume that, for each \( k = 1, \ldots, m \), the Hermite-Birkhoff interpolation subproblems
\[ P_k^{(j)} [f] (x_i) = f^{(j)} (x_i), \quad x_i \in F_k, \ j \in \mathcal{J}_i, \]  
(13)
have a unique solution in their appropriate polynomial spaces \( P_{q_k} \), where \( q_k = \sum_{x_i \in F_k} \#(\mathcal{J}_i) - 1. \) In order to test the solvability of the problems (12) and (13) one can use the necessary Polya’s condition [4] or the sufficient condition given by Atkinson-Sharma theorem [8] and, in case of existence, the solution can be computed by solving a linear system of \( q_k \) linear equations in \( q_k \) unknowns. In fact, there are not explicit expression for the polynomials \( P_k (x) \) even though many algorithm for calculating them are known [9, 10, 11].

The use of determinants allows also to give an explicit expression for the remainder term
\[ R_k [f] (x) = f (x) - P_k [f] (x) \]
which is obtained in [1] by using the Peano’s kernel Theorem [12] and from which a bound can be computed. Nonetheless, this formulation, for its extreme generality is not useful for our purpose. Therefore we state the following

**Theorem 1.** Let us suppose, for each \( k = 1, \ldots, m \), \( F_k = \{x_{k_1}, x_{k_2}, \ldots, x_{k_l}\} \) with \( x_{k_1} < x_{k_2} < \cdots < x_{k_l} \). Let us set \( q_{\max} = \max_k q_k \) and let \( \Omega \) be a closed interval containing \( X \). If \( f \in C^{q_{\max}+1} (\Omega) \), then
\[ |f (x) - P_k [f] (x)| \leq (1 + ||P_k||_\infty) \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)} \left| f^{(q_k+1)} (\xi_k) \right|, \]  
(14)
where \( \min (x, x_{k_1}) < \xi_k < \max (x, x_{k_l}) \).
Proof. Since we require the uniqueness of the solution of the local Hermite-Birkhoff interpolation problem \( (13) \), the operator \( P_k \) is a projector \( 13 \) on the polynomial space \( P_{q_k}^x \), that is it reproduces polynomials up to the degree \( q_k \). Let denote by \( H_k \{ f \} (x) \) the Hermite interpolation polynomial on the nodes of \( F_k \) with \( \#(J_i) \) interpolation conditions on each node \( x_i \in F_k \). Then we have

\[
|f(x) - P_k \{ f \} (x)| \leq |f(x) - H_k \{ f \} (x)| + |H_k \{ f \} (x) - P_k \{ f \} (x)|
\]

\[
\leq |f(x) - H_k \{ f \} (x)| + |P_k H_k \{ f \} (x) - P_k \{ f \} (x)|
\]

\[
\leq (1 + |P_k|) |f(x) - H_k \{ f \} (x)|.
\]

On the other hand, it is well known a Cauchy representation for the remainder term in Hermite interpolation \( 12 \)

\[
f(x) - H_k \{ f \} (x) = \prod_{x_i \in F_k} (x - x_i)^{\#(J_i)} \frac{f^{(q_k+1)}(\xi_k)}{(q_k+1)!}
\]

where \( \min(x, x_{k_1}) < \xi_k < \max(x, x_{k_l}) \). Inequality \( 14 \) follows by taking the modulus of both sides of \( 16 \) and by substituting it in \( 15 \). \( \square \)

4. Multinode global interpolation operator

As soon as we have provided a solution for all local Hermite-Birkhoff interpolation problems, we define the multinode global interpolation operator by

\[
M_{\mu} \{ f, F \} (x) = \sum_{k=1}^{m} B_{\mu, k}(x) P_k \{ f \} (x)
\]

where \( P_k \{ f \} (x) \) is the polynomial solution of the Hermite-Birkhoff interpolation problem on \( F_k \). The operator \( M_{\mu} \{ f, F \} (x) \) has remarkable properties. Firstly, it reproduces polynomials up to a certain degree as specified in the following Proposition.

Proposition 4. The operator \( M_{\mu} \{ f, F \} \) reproduces polynomials up to the degree \( q_{\min} = \min_k q_k \).

Proof. Since the Hermite-Birkhoff interpolation subproblems \( 13 \) have a unique solution in the polynomial spaces \( P_{q_k}^x \), \( P_k \{ f \} (x) \) reproduces polynomials up to
the degree $q_k$ for $k = 1, \ldots, m$. Moreover, the basis functions $B_{\mu,k}(x)$ are a partition of unity and therefore, for each $p \in \mathcal{P}_{\mu}^{q_{\min}}$

$$M_{\mu}[p, \mathcal{F}](x) = \sum_{k=1}^{m} B_{\mu,k}(x) P_k[p](x)$$

$$= \sum_{k=1}^{m} B_{\mu,k}(x)p(x) = p(x).$$

\[ \blacksquare \]

**Remark 1.** If the initial problem has a unique polynomial solution, by setting $\mathcal{F} = \{X\}$, $M_{\mu}[f, \mathcal{F}](x)$ coincides with that polynomial solution.

Secondly, the operator $M_{\mu}[f, \mathcal{F}]$ is an interpolation operator as specified in the following Theorem.

**Theorem 2.** The operator $M_{\mu}[f, \mathcal{F}]$ interpolates the functional data

$$M_{\mu}[f, \mathcal{F}](x_i) = f(x_i), \quad \text{for each } i \text{ such that } 0 \in \mathcal{J}_i$$

and, if $\mathcal{F}$ is a partition of $X$ (i.e. $F_\alpha \cap F_\beta = \emptyset$ for each $\alpha \neq \beta$) the operator $M_{\mu}[f, \mathcal{F}]$ interpolates all data used in its definition, i.e.

$$M_{\mu}^{(j)}[f, \mathcal{F}](x_i) = f^{(j)}(x_i), \quad \text{for each } k = 1, \ldots, m, x_i \in F_k, j \in \mathcal{J}_i.$$

**Proof.** Let be $x_i \in X$ such that $0 \in \mathcal{J}_i$, we have

$$M_{\mu}[f, \mathcal{F}](x_i) = \sum_{k=1}^{m} B_{\mu,k}(x_i) P_k[f](x_i) = \sum_{k=1}^{m} B_{\mu,k}(x_i)f(x_i) = f(x_i).$$

Let be $\ell \in \{1, \ldots, q_{\min}\}$ and $\mu > q_{\min} + 1$. By differentiating $\ell$ times (17) we get, by the Leibniz rule for differentiation

$$M_{\mu}^{(\ell)}[f, \mathcal{F}](x) = \sum_{k=1}^{m} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B_{\mu,k}^{(\ell-i)}(x) P_k^{(i)}[f](x)$$

$$= \sum_{k=1}^{m} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B_{\mu,k}^{(\ell-i)}(x) P_k^{(i)}[f](x) + \sum_{k=1}^{m} B_{\mu,k}(x) P_k^{(\ell)}[f](x).$$

Let be $x_i \in X$ and $\ell \in \mathcal{J}_i$. Since $\mathcal{F}$ is a partition of $X$ then $K_i = \{\kappa\}$. By properties (6) and (7) we have

$$\sum_{k=1}^{m} B_{\mu,k}(x_i) P_k^{(\ell)}[f](x_i) = \sum_{k=1}^{m} B_{\mu,k}(x_i) P_k^{(\ell)}[f](x_i) + B_{\mu,\kappa}(x_i) P_{\kappa}^{(\ell)}[f](x_i)$$

$$= f^{(\ell)}(x_i).$$
On the other hand, by properties (9) and (10) we have
\[
\sum_{k=1}^{m} B^{(\ell-i)}_{\mu,k}(x_i) P_k^{(i)}[f](x_i) = \sum_{k=1}^{m} B^{(\ell-i)}_{\mu,k}(x_i) P_k^{(i)}[f](x_i) + B^{(\ell-i)}_{\mu,k}(x_i) P_k^{(i)}[f](x_i)
\]
\[
= 0
\]
for each \(i = 0, \ldots, \ell - 1\).

Let us observe that the operator \(M_{\mu}[f,F]\) could not interpolate all derivative data at some \(x_\kappa\) if \(\sharp(K_\kappa) > 1\) and the sequence of indices in \(J_\kappa\) is broken.

**Example 1.** For example, let us assume
\[\sharp(K_\kappa) = 2,\quad F_\alpha \cap F_\beta = \{x_\kappa\},\quad J_\kappa = \{0, 2, \ldots, \ell - 1, \ell\}, \ell \geq 2\]
and
\[B^{(\ell-1)}_{\mu,\alpha}(x_\kappa) P_\alpha'[f](x_\kappa) + B^{(\ell-1)}_{\mu,\beta}(x_\kappa) P_\beta'[f](x_\kappa) \neq 0.\]
We note that
\[P_\alpha'[f](x_\kappa) \neq P_\beta'[f](x_\kappa)\]
since property (10). From
\[P_\alpha^{(\ell)}[f](x_\kappa) = P_\beta^{(\ell)}[f](x_\kappa) = f^{(\ell)}(x_\kappa),\]
by properties (9) and (10), easily follows
\[
\sum_{k=1}^{m} B_{\mu,k}(x_\kappa) P_k^{(i)}[f](x_\kappa) = B_{\mu,\alpha}(x_\kappa) f^{(\ell)}(x_\kappa) + B_{\mu,\beta}(x_\kappa) f^{(\ell)}(x_\kappa)
\]
\[
= f^{(\ell)}(x_\kappa).
\]
On the other hand,
\[
\sum_{k=1}^{m} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B^{(\ell-i)}_{\mu,k}(x_\kappa) P_k^{(i)}[f](x_\kappa)
\]
\[
= \sum_{i=0}^{\ell-1} \binom{\ell}{i} \left( B^{(\ell-i)}_{\mu,\alpha}(x_\kappa) P_\alpha^{(i)}[f](x_\kappa) + B^{(\ell-i)}_{\mu,\beta}(x_\kappa) P_\beta^{(i)}[f](x_\kappa) \right)
\]
by property (9). Let us fix our attention to the right hand side of previous equality. For each \( \iota \in J \), we get

\[
B^{(\ell - 1)}(x) P^{(\ell - 1)} P^{[f]}(x) + B^{(\ell - 1)}(x) P^{[f]}(x) = \left( B^{(\ell - 1)}(x) + B^{(\ell - 1)}(x) \right) P^{(i)}(x) = 0
\]

by property (10), but

\[
B^{(\ell - 1)}(x) P^{[f]}(x) + B^{(\ell - 1)}(x) P^{[f]}(x) \neq 0
\]

and consequently

\[
M^{(\ell)} [f, F] (x) \neq f^{(\ell)} (x).
\]

In order to avoid this trouble, we proceed as follows. For each \( \kappa = 1, \ldots, n \) let be \( \nu = \sharp (K_\kappa) \) and \( F_{\alpha_1}, \ldots, F_{\alpha_{\nu_k}} \) the subset of \( X \) which contain \( x_\kappa \). As above, let us denote by \( P_{\alpha_1} [f], \ldots, P_{\alpha_{\nu_k}} [f] \) the polynomial solutions of the Hermite-Birkhoff interpolation problems on \( F_{\alpha_1}, \ldots, F_{\alpha_{\nu_k}} \) respectively. For all \( j = 0, 1, \ldots, \max (J_\kappa) \) we set

\[
\tilde{f}^{(j)}(x_\kappa) = \frac{1}{\nu_k} \left( P_{\alpha_1}^{(j)} [f](x_\kappa) + \cdots + P_{\alpha_{\nu_k}}^{(j)} [f](x_\kappa) \right)
\]

and we note that

\[
\tilde{f}^{(j)}(x_\kappa) = f^{(j)}(x_\kappa)
\]

as soon as \( j \in J_\kappa \). For each \( k = 1, \ldots, m \) we call the Hermite interpolation problem

\[
\tilde{P}_k^{(j)} [f](x_i) = \tilde{f}^{(j)}(x_i), \quad x_i \in F_k, \; j = 0, 1, \ldots, \max (J_i),
\]

hermitian completion of the Hermite-Birkhoff interpolation problem \( [13] \). It is well known that each interpolation problem \( [24] \) has a unique solution \( \tilde{P}_k[f](x) \) in the polynomial space \( P^d_k, d_k = \sharp (F_k) + \sum_{x_i \in F_k} \max (J_i) - 1 \), for which there are explicit formulas in Lagrange or Newton form (see \( [14, 15, 16] \) and the references therein). Nevertheless, as soon as \( \nu_k > 1 \) and at least two among \( P_{\alpha_1}^{(j)} [f](x_\kappa), \ldots, P_{\alpha_{\nu_k}}^{(j)} [f](x_\kappa) \) are different, we get \( q_k < d_k \); in this case, if \( p \in P^q_k \) is a generic polynomial, then \( \tilde{P}_k[p] \) is different from \( p \), since, by \( [19] \),
we have completed the lacunary data using solutions of several interpolation problems. In fact we have

\[ \tilde{q}_k = \text{dex} \left( \tilde{P}_k [\cdot] \right) = \min_{j=0,1, \ldots, \text{max}(J)} \left\{ \text{dex} \left( P_{\alpha_j} [\cdot] \right) \right\} \]

and the proof is straightforward. Consequently, \( q_k \leq \tilde{q}_k \). Results on the bound for the remainder term

\[ \tilde{R}_k [f] (x) = f(x) - \tilde{P}_k [f] (x) \]

can be obtained similarly to Theorem 3, taking into account that \( \text{dex} \left( \tilde{P}_k [\cdot] \right) = \tilde{q}_k \).

**Theorem 3.** Let us suppose, for each \( k = 1, \ldots, m \), \( F_k = \{ x_{k_1}, x_{k_2}, \ldots, x_{k_l} \} \) with \( x_{k_1} < x_{k_2} < \cdots < x_{k_l} \). Let us set \( q_{\text{max}} = \max_k q_k \) and let \( \Omega \) be a closed interval containing \( X \). If \( f \in C^{q_{\text{max}}+1} (\Omega) \), then

\[
\left| f(x) - \tilde{P}_k [f] (x) \right| \leq \left( 1 + \frac{\prod_{x_i \in F_k} (x-x_i)^{\#(J_i)-\#(S_i)}}{(\tilde{q}_k + 1)!} \right) f(\tilde{q}_k+1) (\xi_k),
\]

where \( \min(x, x_k) < \xi_k < \max(x, x_k) \) and \( S_i \subset J_i, i = 1, \ldots, n \), such that

\[
\sum_{x_i \in F_k} \#(J_i) - \sum_{x_i \in F_k} \#(S_i) = \tilde{q}_k + 1.
\]

**Proof.** Since \( \text{dex} \left( \tilde{P}_k [\cdot] \right) = \tilde{q}_k \) the operator \( \tilde{P}_k \) is a projector on the polynomial space \( P_k^{\tilde{q}_k} \), that is it reproduces polynomials up to the degree \( \tilde{q}_k \). Let denote by \( \tilde{H}_k [f] (x) \) the Hermite interpolation polynomial on the nodes of \( F_k \) with \( \#(J_i) - \#(S_i) \) interpolation conditions on each node \( x_i \in F_k \). Then we have

\[
\left| f(x) - \tilde{P}_k [f] (x) \right| \leq \left| f(x) - \tilde{H}_k [f] (x) \right| + \left| \tilde{H}_k [f] (x) - \tilde{P}_k [f] (x) \right|
\]

\[
\leq \left| f(x) - \tilde{H}_k [f] (x) \right| + \left| \tilde{P}_k \tilde{H}_k [f] (x) - \tilde{P}_k [f] (x) \right|
\]

\[
\leq \left( 1 + \left\| \tilde{P}_k \right\| \right) \left| f(x) - \tilde{H}_k [f] (x) \right|.
\]

Therefore

\[
f(x) - \tilde{H}_k [f] (x) = \frac{\prod_{x_i \in F_k} (x-x_i)^{\#(J_i)-\#(S_i)}}{(\tilde{q}_k + 1)!} f(\tilde{q}_k+1) (\xi_k)
\]

where \( \min(x, x_k) < \xi_k < \max(x, x_k) \).
Example 2. Let be $X = \{-1, 0, 1\}$ with interpolation conditions as in (1) or Figure [1]. In this case $F_1 = \{-1, 0\}$, $F_2 = \{0, 1\}$, $P_1[f](x) = f(-1) + (1 + x)f'(0)$ $P_2[f](x) = f(1) + (-1 + x)f'(0)$

and

$\tilde{P}_1[f](x) = \frac{f(-1)+f(1)}{2} + f'(0)x + 3\left(\frac{f(1)-f(-1)}{2} + f'(0)\right)x^2$ $+ (f(-1) + f(1) + 2f'(0))x^3$

$\tilde{P}_2[f](x) = \frac{f(-1)+f(1)}{2} + f'(0)x + 3\left(\frac{f(1)-f(-1)}{2} - f'(0)\right)x^2$ $+ (f(-1) - f(1) + 2f'(0))x^3$.

Let be $f(x) = \sin(x)$. In Fig. 3 we display polynomials $P_1[f]$, $P_2[f]$ and their hermitian completions $\tilde{P}_1[f]$ and $\tilde{P}_2[f]$ respectively. In Fig. 4 we display the absolute values of the errors $e[f,F](x) = f(x) - M_2[f,F](x)$ and $\tilde{e}[f,F](x) = f(x) - \tilde{M}_2[f,F](x)$ in the interval $[-1, 1]$, where

$M_2[f,F](x) = B_{2,1}(x)P_1[f](x) + B_{2,2}(x)P_2[f](x)$,

and

$\tilde{M}_2[f,F](x) = B_{2,1}(x)\tilde{P}_1[f](x) + B_{2,2}(x)\tilde{P}_2[f](x)$.

As we can see, at least in this case, the inequality $|e[f,F](x)| \leq |\tilde{e}[f,F](x)|$ holds for all $x \in [-1, 1]$.

We set

$\tilde{M}_\mu[f,F](x) = \sum_{k=1}^{m} B_{\mu,k}(x)\tilde{P}_k[f](x)$. 


Figure 3: Polynomials $P_1[f]$ (blue) and $P_2[f]$ (red), on the left, and their hermitian completions $\tilde{P}_1[f]$ and $\tilde{P}_2[f]$, on the right, respectively.
Figure 4: Comparison between the absolute values of the errors $e[f, F](x)$ (blue) and $\tilde{e}[f, F](x)$ (red) for the function $f(x) = \sin(x)$ in the interval $[-1, 1]$.

The operator $\tilde{M}_\mu[F]$ preserves the reproducing polynomial property of $M_\mu[F]$ and, in addition, interpolates all derivative data. In fact we have

**Proposition 5.** The operator $\tilde{M}_\mu[f, F]$

(a) reproduces polynomials up to the degree $q_{\min} = \min_k q_k$.

(b) interpolates all data used in its definition, i.e.

$$\tilde{M}_\mu^{(j)}[f, F](x_i) = f^{(j)}(x_i), \quad \text{for each } k = 1, \ldots, m, \ x_i \in F_k, \ j \in J_i.$$

**Proof.** (a) Let $p \in P_{x}^{q_{\min}}$. Since the Hermite-Birkhoff interpolation subproblems (13) have a unique solution in the polynomial spaces $P_{x}^{q_k}$ it results $P_k[p] = p$ for each $k = 1, \ldots, m$. Setting (13) now becomes

$$\tilde{p}^{(j)}(x_k) = \frac{1}{\nu_k} \left( p^{(j)}(x_k) + \cdots + p^{(j)}(x_k) \right)$$

$$= p^{(j)}(x_k)$$

and consequently the hermitian completion of the Hermite-Birkhoff interpolation problem (13)

$$\tilde{P}_k^{(j)}[p](x_i) = \tilde{p}^{(j)}(x_k), \quad x_i \in F_k, \ j = 0, 1, \ldots \max(J_i),$$

has solution $\tilde{P}_k[p] = p$ since $q_k \leq d_k$. The thesis follows since the basis functions $B_{\mu,k}(x)$ are a partition of unity.
(b) Let be $\ell \in \{0, 1, \ldots, q\}$ and $\mu > q + 1$. By differentiating $\ell$ times (17) we get, by the Leibniz rule for differentiation

$$M_\mu^{(\ell)} [f, F](x) = \sum_{k=1}^{m} \sum_{i=0}^{\ell} \binom{\ell}{i} B_{\mu,k}^{(\ell-i)}(x) \tilde{P}_k^{(i)}[f](x)$$

$$= \sum_{k=1}^{m} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B_{\mu,k}^{(\ell-i)}(x) \tilde{P}_k^{(i)}[f](x) + \sum_{k=1}^{m} B_{\mu,k}(x) \tilde{P}_k^{(\ell)}[f](x).$$

Let be $x_i \in X, F_{\alpha_1}, \ldots, F_{\alpha_{\nu_i}}$ the subset of $X$ which contain $x_i$ and $\ell \in J_i$. By properties (6) and (7) we have

$$\sum_{k=1}^{m} B_{\mu,k}(x_i) \tilde{P}_k^{(\ell)}[f](x_i) = \sum_{k=1}^{m} B_{\mu,k}(x_i) \tilde{P}_k^{(\ell)}[f](x_i)_{\neq \alpha_1, \ldots, \alpha_{\nu_i}}$$

$$= \sum_{k=1}^{m} 0 + \tilde{f}^{(\ell)}(x_i) \sum_{k=1}^{m} B_{\mu,k}(x_i)$$

$$= f^{(\ell)}(x_i)$$

since (21) and (20). On the other hand, for each $i = 0, \ldots, \ell - 1$ we have

$$\sum_{k=1}^{m} B_{\mu,k}(x_i) \tilde{P}_k^{(\ell-i)}[f](x_i) = \sum_{k=1}^{m} B_{\mu,k}(x_i) \tilde{P}_k^{(\ell-i)}[f](x_i)_{\neq \alpha_1, \ldots, \alpha_{\nu_i}}$$

$$+ \sum_{k=1}^{m} B_{\mu,k}(x_i) \tilde{P}_k^{(\ell-i)}[f](x_i)_{\neq \alpha_1, \ldots, \alpha_{\nu_i}}$$

$$= \sum_{k=1}^{m} 0 + \tilde{f}^{(\ell)}(x_i) \sum_{k=1}^{m} B_{\mu,k}(x_i)$$

$$= 0$$

since (9), (10) and (21). ■

In the next sections we will discuss the approximation order of operators $M_\mu [f, F](x), \tilde{M}_\mu [f, F](x)$ and we will compare their accuracies of approximation in several interesting cases.
5. The approximation order

Let us now study the approximation order of the operators $M_\mu [f, \mathcal{F}] (x)$ and $\tilde{M}_\mu [f, \mathcal{F}] (x)$. To this aim, let $||\cdot||$ be the maximum norm and $R_\rho (y) = \{ x \in \mathbb{R} : |x - y| \leq \rho \}$ be the closed interval of center $y$ and size $2\rho$. We define

$$h = \inf \{ \rho > 0 : \forall x \in \Omega \ \exists F_k \in \mathcal{F} : R_\rho (x) \cap F_k \neq \emptyset \} \quad (22)$$

and

$$L = \inf \{ l > 0 : \forall F_k \in \mathcal{F} \ \exists x \in \Omega : F_k \subset R_{lh} (x) \} , \quad (23)$$

$$M = \sup_{x \in \Omega} \sharp \{ F_k \in \mathcal{F} : R_h (x) \cap F_k \neq \emptyset \} , \quad (24)$$

$$N = \max_i \sharp \{ F_k \in \mathcal{F} : x_i \in F_k \} = \max_i \sharp \{ K_i \} . \quad (25)$$

A small value of $h$ corresponds to a rather uniform distribution of nodes, but does not exclude the presence of large $F_k$ (note that $Lh$ is half the diameter of the largest $F_k$). A large value of $M$ corresponds to clustered $F_k$ and $N$ is the maximum number of $F_k$ with at least a common node ($N = 1$ means that $\mathcal{F}$ is a partition of $X$). Moreover, we set

$$N_k = \sum_{x_i \in F_k} \# (\mathcal{J}_i) = q_k + 1,$$

and

$$C_{F_{\max}} = \max_k \sharp (F_k) ,$$

$$C_{F_{\min}} = \min_k \sharp (F_k) ,$$

$$C_{N_{\max}} = \max_k N_k ,$$

$$C_{N_{\min}} = \min_k N_k .$$

**Theorem 4.** Let $\Omega$ be an interval that contains $X$, $f \in C^{q_{\max}+1} (\Omega)$, $\mu > \frac{1 + C_{N_{\max}}}{C_{F_{\min}}} \frac{C_{F_{\max}}}{C_{F_{\min}}} \phi_{\max}$ and $\sharp (F_k) = \text{const}$ for each $k$. Then

$$| f (x) - M_\mu [f, \mathcal{F}] (x) | \leq CMh^{C_{N_{\max}} / \phi_{\max}}$$
for any $x \in \Omega$, with $C$ a positive constant which depends on $F_k$ and $\mu$ and

$$\phi_{\text{max}} = \max_{j=0, \ldots, \phi_{\text{max}}} \|f^{(j)}\|_{\infty}.$$

**Proof.** For $x \in \Omega$ let

$$Q_h (y) = \{ x \in \mathbb{R} : y - h < x \leq y + h \}$$

be the half-open interval with center $y$ and size $2h$. Let $T_j = T_{j,h} (x)$ be the half-open annulus with center $x$ and radius $2hj$ defined by

$$T_j = Q_h (x - 2hj) \cup Q_h (x + 2hj), \quad j = 0, \ldots, N.$$

Note that $T_0 = Q_h (x)$ and that

$$\Omega \subset \bigcup_{j=0}^{N} T_j, \quad N = \left[ \frac{\text{diam}(\Omega)}{2h} \right] + 1.$$

By settings (22) and (24) we have

$$1 \leq \sharp (X \cap T_{j,h}) \leq 2M.$$

$T_0 = Q_h (x)$ contains at least a node by setting (22); therefore, for each $F_k$ with at least a node in $T_0$, we have

$$\prod_{x_i \in F_k} |x - x_i| \leq (Lh)^{\sharp(F_k)}$$

since (23). Let us consider now the sets $F_k$ with at least a node in $T_1$ and no nodes in $T_0$; we have

$$h^{\sharp(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq [(L + 3)h]^{\sharp(F_k)},$$

while for the sets $F_k$ with at least a node in $T_2$ and no nodes in $T_1 \cup T_0$ we have

$$(3h)^{\sharp(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq [(L + 5)h]^{\sharp(F_k)}$$

and in general, for the sets $F_k$ with at least a node in $T_j$ and no nodes in $T_{j-1} \cup \cdots \cup T_0$,

$$[(2j - 1)h]^{\sharp(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq [(L + 2j + 1)h]^{\sharp(F_k)}.$$
Let us now turn to the approximation error
\[ e(x) = |f(x) - M_{\mu}[f, F](x)| \]
of the multinode global operator at \( x \). By (14) and the fact that the basis function \( B_{k,\mu}(x) \) are non-negative and form a partition of unity,
\[ e(x) \leq \sum_{k=1}^{m} B_{k,\mu}(x)f(x) - \sum_{k=1}^{m} B_{k,\mu}(x)P_{k}[f](x) \leq \sum_{k=1}^{m} |f(x) - P_{k}[f](x)| B_{k,\mu}(x). \]
Using Proposition 1 and (4) we then get
\[ e(x) \leq \sum_{k=1}^{m} (1 + \|P_{k}\||) \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})} \frac{|x - x_{i}|^{\#(J_{i})}}{(q_{k} + 1)!} f(q_{k} + 1) (\xi_{k}) \prod_{x_{i} \in F_{k}} |x - x_{j}|^{-\mu} \sum_{l=1}^{m} \prod_{x_{j} \in F_{l}} |x - x_{j}|^{-\mu} \]
(28)
Let \( F_{k_{\min}} \in F \) be the subset such that
\[ \prod_{x_{i} \in F_{k_{\min}}} |x - x_{i}| = \min_{k} \prod_{x_{i} \in F_{k}} |x - x_{i}|. \]
Then
\[ \prod_{x_{i} \in F_{k_{\min}}} |x - x_{i}| \leq (Lh)^{2(F_{0})} \]
since at least a set \( F_{k_{0}} \) has a node in \( T_{0} \). Finally, we have to bound, for each \( k = 1, \ldots, m, \)
\[ \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})}. \]
For each \( F_{k} \) with at least a node in \( T_{0} \) we then have
\[ \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})} \leq \prod_{x_{i} \in F_{k}} (Lh)^{2(F_{0})} \leq (Lh)^{N_{k}}; \]
and for each \( F_{k} \) with no nodes in \( T_{0} \cup \cdots \cup T_{j-1} \) and at least a node in \( T_{j}, \)
\[ \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})} \leq \prod_{x_{i} \in F_{k}} ((L + 2j + 1)h)^{2(F_{0})} \leq ((L + 2j + 1)h)^{N_{k}}. \]
Then we get
\[ e(x) \leq \sum_{k=1}^{m} (1 + \|P_{k}\||) \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})} \frac{|x - x_{i}|^{\#(J_{i})}}{(q_{k} + 1)!} f(q_{k} + 1) (\xi_{k}) \prod_{x_{i} \in F_{k}} |x - x_{i}|^{-\mu} \]
\[ \leq \prod_{x_{i} \in F_{k_{\min}}} |x - x_{i}|^{\mu} \sum_{k=1}^{m} (1 + \|P_{k}\||) \prod_{x_{i} \in F_{k}} |x - x_{i}|^{\#(J_{i})} \frac{|x - x_{i}|^{\#(J_{i})}}{(q_{k} + 1)!} f(q_{k} + 1) (\xi_{k}) \prod_{x_{i} \in F_{k}} |x - x_{i}|^{-\mu}. \]
We denote by $I_j$ the set of subsets $F_k$ with at least a node in $T_j$ and no nodes in $T_0 \cup \cdots \cup T_{j-1}$. For construction, $\bigcup_{j=0}^{N} I_j = F$ and $\bigcap_{j=0}^{N} I_j = \emptyset$. Moreover, we set

$$P_{\text{max}} = \max_k ||P_k||,$$

$$\phi_{\text{max}} = \max_{j=0,\ldots,q_{\text{max}}} ||f(j)||_{\infty}$$

then by bounding (28) we have

$$e(x) \leq \left(1 + P_{\text{max}} \right) \frac{\phi_{\text{max}}}{(q_{\text{min}} + 1)!} \prod_{x_i \in F_{\text{min}}} |x - x_i|^{\mu} \sum_{k=1}^{m} \prod_{x_i \in F_k} |x - x_i|^{|J_k|} \prod_{x_i \in F_k} |x - x_i|^{-\mu}$$

$$\leq \left(1 + P_{\text{max}} \right) \frac{\phi_{\text{max}}}{(q_{\text{min}} + 1)!} \prod_{x_i \in F_{\text{min}}} |x - x_i|^{|J_k|} \left( \prod_{x_i \in F_k} |x - x_i|^{-\mu} \right) \left( \prod_{x_i \in F_k} |x - x_i|^{-\mu} \right)$$

$$+ \sum_{j=1}^{N} \sum_{F_k \in I_j} \prod_{x_i \in F_k} |x - x_i|^{|J_k|} \left( \prod_{x_i \in F_k} |x - x_i|^{-\mu} \right) \right).$$

For subsets $F_k \in I_0$ it results

$$\prod_{x_i \in F_{\text{min}}} |x - x_i| \prod_{x_i \in F_k} |x - x_i| \leq 1$$

while if $F_k \in I_j$, $j \geq 1$ then

$$\frac{\prod_{x_i \in F_{\text{min}}} |x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \leq \frac{(Lh)^{2(F_{\text{min}})}}{[(2j - 1) h]^{2(F_k)}} = \frac{L^{2(F_{\text{min}})}}{(2j - 1)^{2(F_k)}} h^{2(F_{\text{min}}) - 2(F_k)}$$

19
Let us assume that \( C \) converges for \( \#(J_i) \).

\[
e(x) \leq \left( \frac{1 + P_{\text{max}}}{q_{\text{min}} + 1} \right) \phi_{\text{max}} \left( \sum_{F_k \in Z_0} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i)} \right)
\]

\[
+ \sum_{j=1}^{N} \sum_{F_k \in Z_0} \prod_{x_i \in F_k} (|x - x_i|^{\#(J_i)} \left( \frac{L^j(F_k)}{(2j - 1)h(F_k)} h(F_k) - h(F_k) \right)^{\mu})
\]

\[
\leq \left( \frac{1 + P_{\text{max}}}{q_{\text{min}} + 1} \right) \phi_{\text{max}} \left( \sum_{F_k \in Z_0} (L h_n)^{N_k} \right)
\]

\[
+ \sum_{j=1}^{N} \sum_{F_k \in Z_0} (L + 2j + 1)^{N_k} h_n \left( \frac{L^j(F_k)}{(2j - 1)h(F_k)} h(F_k) - h(F_k) \right)^{\mu})
\]

\[
\leq \left( \frac{1 + P_{\text{max}}}{q_{\text{min}} + 1} \right) \phi_{\text{max}} h^{C_{\text{min}}} \left( \sum_{F_k \in Z_0} L^{C_{\text{max}}} \right)
\]

\[
+ \sum_{j=1}^{N} \sum_{F_k \in Z_0} (L + 2j + 1)^{C_{\text{max}}} \left( \frac{L^{C_{\text{max}}}}{(2j - 1)^{\mu_{\text{min}}}} \right) h_n \left( h(F_k) - h(F_k) \right)^{\mu}
\]

Let us assume that \( h(F_k) = \text{const} \) for each \( k \), then

\[
e(x) \leq \left( \frac{1 + P_{\text{max}}}{q_{\text{min}} + 1} \right) \phi_{\text{max}} h^{C_{\text{min}}} \left( M L^{C_{\text{max}}} + M \sum_{j=1}^{N} (L + 2j + 1)^{C_{\text{max}}} \left( \frac{L^{C_{\text{max}}}}{(2j - 1)^{\mu_{\text{min}}}} \right)^{\mu} \right)
\]

\[
\leq \left( \frac{1 + P_{\text{max}}}{q_{\text{min}} + 1} \right) \phi_{\text{max}} h^{C_{\text{min}}} M \left( L^{C_{\text{max}}} + M L^{C_{\text{max}}} \left( \sum_{j=1}^{N} (L + 2j + 1)^{C_{\text{max}}} \left( \frac{L^{C_{\text{max}}}}{(2j - 1)^{\mu_{\text{min}}}} \right)^{\mu} \right) \right)
\]

Let us consider the series

\[
\sum_{j=1}^{\infty} \frac{(L + 2j + 1)^{C_{\text{max}}}}{(2j - 1)^{\mu_{\text{min}}}} \approx \sum_{j=1}^{\infty} \frac{(2j)^{C_{\text{max}}}}{(2j - 1)^{\mu_{\text{min}}}} = \sum_{j=1}^{\infty} \frac{1}{(2j)^{\mu_{\text{min}} - C_{\text{max}}}}
\]

it converges for \( \mu^{C_{\text{min}}} - C_{\text{max}} > 1 \), i.e. for \( \mu > \frac{1 + C_{\text{max}}}{C_{\text{min}}} \) the operator \( M_{\mu} [f, F] \) has approximation order \( O(h^{C_{\text{min}}}) \).

Let us now study the approximation order of the operator \( \tilde{M}_{\mu} [f, F] (x) \). To this aim, we set

\[
\tilde{N}_k = \sum_{x_i \in F_k} \#(J_i) - \sum_{x_i \in F_k} \#(S_i) = \tilde{q}_k + 1
\]

and

\[
C_{\tilde{N}_{\text{max}}} = \max_k \tilde{N}_k,
\]

\[
C_{\tilde{N}_{\text{min}}} = \min_k \tilde{N}_k.
\]
**Theorem 5.** Let \( \Omega \) be an interval that contains \( X \), \( f \in C^{q_{\max}+1}(\Omega) \), \( \mu > \frac{1+C_{N_{\max}}}{\epsilon F_{\min}} \) and \( \sharp(F_k) = \text{const} \) for each \( k \). Then

\[
|f(x) - \tilde{M}_\mu[f,F](x)| \leq CMhC_{\tilde{N}_{\min}}\phi_{\max}
\]

for any \( x \in \Omega \), with \( C \) a positive constant which depends on \( F_k \) and \( \mu \).

**Proof.** Let be

\[
\tilde{e}(x) = |f(x) - \tilde{M}_\mu[f,F](x)|.
\]

By (17) and the fact that the basis function \( B_{k,\mu}(x) \) are non-negative and form a partition of unity,

\[
\tilde{e}(x) \leq \sum_{k=1}^{m} B_{k,\mu}(x) f(x) - \sum_{k=1}^{m} B_{k,\mu}(x) \tilde{P}_k[f](x) \leq \sum_{k=1}^{m} |f(x) - \tilde{P}_k[f](x)| B_{k,\mu}(x) .
\]

Using Theorem 3 and (4) we then get

\[
\tilde{e}(x) \leq \sum_{k=1}^{m} \left( 1 + \|\tilde{P}_k\|_\infty \right) \prod_{x_i \in F_k} |x - x_i|^{\sharp(J_i)} \frac{|f(\tilde{q}_k+1)(\xi_k)|}{(\tilde{q}_k + 1)!} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \prod_{i=1}^{m} \prod_{x_j \in F_i} |x - x_j|^{-\mu} .
\]

Let \( F_{k_{\min}} \in F \) be the subset such that

\[
\prod_{x_i \in F_{k_{\min}}} |x - x_i| = \min_k \prod_{x_i \in F_k} |x - x_i| .
\]

Then

\[
\prod_{x_i \in F_{k_{\min}}} |x - x_i| \leq (Lh)^{\sharp(F_{k_{\min}})}
\]

since at least a set \( F_{k_{\min}} \) has a node in \( T_0 \). Finally, we have to bound, for each

\[
k = 1, \ldots, m ,
\]

\[
\prod_{x_i \in F_k} |x - x_i|^{\sharp(J_i)} .
\]

For each \( F_k \) with at least a node in \( T_0 \) we then have

\[
\prod_{x_i \in F_k} |x - x_i|^{\sharp(J_i)} \leq \prod_{x_i \in F_k} (Lh)^{\sharp(J_i)} \leq (Lh)^{N_k} ;
\]
and for each $F_k$ with no nodes in $T_0 \cup \cdots \cup T_{j-1}$ and at least a node in $T_j$,
\[
\prod_{x_i \in F_k} |x - x_i|^{\#(J_i)} \leq \prod_{x_i \in F_k} ((L + 2j + 1)h)^{\#(J_i)} \leq ((L + 2j + 1)h)^{N_k}.
\]
Then we get
\[
\tilde{c}(x) \leq \sum_{k=1}^{m} \left(1 + \left\| \tilde{P}_k \right\|_{\infty} \right) \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \leq \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \sum_{k=1}^{m} \left(1 + \left\| \tilde{P}_k \right\|_{\infty} \right) \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \leq \prod_{x_i \in F_k} |x - x_i|^{\mu} \left(\sum_{j=0}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu}\right) \leq \prod_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu}\].

We denote by $\mathcal{I}_j$ the set of subsets $F_k$ with at least a node in $T_j$ and no nodes in $T_0 \cup \cdots \cup T_{j-1}$. For construction, $\cup_{j=0}^{N} \mathcal{I}_j = \mathcal{F}$ and $\cap_{j=0}^{N} \mathcal{I}_j = \emptyset$. Moreover, we set
\[
\tilde{P}_\max = \max_k \left\| \tilde{P}_k \right\|_{\infty},
\]
then by bounding (29) we have
\[
\tilde{c}(x) \leq \left(1 + \frac{\tilde{P}_\max}{(q_{\max} + 1)}\right) \phi_{\max} \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \leq \left(1 + \frac{\tilde{P}_\max}{(q_{\max} + 1)}\right) \phi_{\max} \prod_{x_i \in F_k} |x - x_i|^{\mu} \left(\sum_{j=0}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu}\right) \leq \prod_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu}.\]

For subsets $F_k \in \mathcal{I}_0$ it results
\[
\prod_{x_i \in F_{k_{\min}}} \frac{|x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \leq 1
\]
while if $F_k \in \mathcal{I}_j, j \geq 1$ then
\[
\prod_{x_i \in F_{k_{\min}}} \frac{|x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \leq \frac{(Lh)^2(F_{k_{\min}})}{((2j - 1)h)^2(F_k)} = \frac{L^2(F_{k_{\min}})}{(2j - 1)^2(F_k)} h^2(F_{k_{\min}}) - h(F_k).\]
Let us consider the series

\[ \tilde{c}(x) \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} \left( \sum_{F_k \in I_0} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \right) \]

+ \sum_{j=1}^{N} \sum_{F_k \in I_j} \prod_{x_i \in F_k} |x - x_i|^{\#(J_i) - \#(S_i)} \left( \frac{L^2 h_{F_0}}{(2j - 1)!} h_t(F_{x_0}) - t(F_{x_0}) \right)^{\mu} \]

\[ \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} \left( \sum_{F_k \in I_0} (L h) \bar{N}_k \right) \]

+ \sum_{j=1}^{N} \sum_{F_k \in I_j} (L + 2j + 1) \bar{N}_k \left( \frac{L^2 h_{F_0}}{(2j - 1)!} h_t(F_{x_0}) - t(F_{x_0}) \right)^{\mu} \]

\[ \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} h^{C_{\text{max}}} \left( \sum_{F_k \in I_0} L^{C_{\text{max}}} \right) \]

\[ + \sum_{j=1}^{N} \sum_{F_k \in I_j} (L + 2j + 1) C_{\text{max}} \left( \frac{L^{C_{F_{\text{max}}}}}{(2j - 1)!} F_{\text{min}} \right) \left( h_t(F_{x_0}) - t(F_{x_0}) \right)^{\mu} \]

Let us assume that \( \mathcal{a}(F_k) = \text{const} \) for each \( k \), then

\[ \tilde{c}(x) \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} h^{C_{\text{max}}} \left( \sum_{F_k \in I_0} L^{C_{\text{max}}} \right) \]

\[ + \sum_{j=1}^{N} \sum_{F_k \in I_j} (L + 2j + 1) C_{\text{max}} \left( \frac{L^{C_{F_{\text{max}}}}}{(2j - 1)!} F_{\text{min}} \right) \left( h_t(F_{x_0}) - t(F_{x_0}) \right)^{\mu} \]

\[ \tilde{c}(x) \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} h^{C_{\text{min}}} M \left( L^{C_{\text{max}}} + M \sum_{j=1}^{N} (L + 2j + 1) C_{\text{max}} \right) \]

\[ \leq \left( 1 + \frac{\bar{P}_{\text{max}}}{\bar{q}_{\text{min}} + 1} \right) \Phi_{\text{max}} h^{C_{\text{min}}} \left( L^{C_{\text{max}}} + (L^{C_{F_{\text{max}}}})(2j - 1)! F_{\text{min}} \right)^{\mu} \sum_{j=1}^{N} \left( L + 2j + 1 \right) C_{\text{max}} \]

Let us consider the series

\[ \sum_{j=1}^{\infty} \frac{(L + 2j + 1) C_{\text{max}}}{(2j - 1)!} F_{\text{min}} \approx \sum_{j=1}^{\infty} \frac{(2j) C_{\text{max}}}{(2j - 1)!} F_{\text{min}} \]

it converges for \( \mu C_{F_{\text{min}}} - C_{\text{max}} > 1 \), i.e. for \( \mu > \frac{1 + C_{\text{max}}}{C_{F_{\text{min}}}} \) the operator \( \widetilde{M}_\mu[f, \mathcal{F}] \) has approximation order \( O \left( h^{C_{\text{min}}} \right) \).

6. Numerical results

To numerically test the approximation order of the Multinode rational interpolation operators predicted by Theorem 3, we carried out a series of exper-
Figure 5: Test functions used in our numerical experiments. The definitions of functions can be found in [17].

Experiments with different sets of equispaced nodes on [0, 1] and test functions (See Figure 5). We report these results in Section 6.1. In Section 6.2 we present numerical result on the approximation accuracy of the Multinode rational interpolation operators.

6.1. Approximation order

Our first series of experiments numerically test the theoretical result on the approximation order in Theorem 4. With this aim we consider different coverings $\mathcal{F}$ of the nodeset $X$ with increasing number of subsets $F_k$ (See Figure 4). For each of the 6 test functions $f_i$ we constructed the multinode rational interpolant $M_\mu [f_i, \mathcal{F}] (x)$ and we determined the maximum approximation error $e_{\text{max}}$ by evaluating $|f_i(x) - M_4 [f_i, \mathcal{F}] (x)|$ at 100,000 random points $x \in [0, 1]$ and recording the maximum value. For the first experiment, the subsets are generated as follows:

1. we fix 3 equispaced points on the interval [0, 1] and we associate to them the Birkhoff data $f(0), f' \left(\frac{1}{2}\right), f(1)$. We consider the subsets $F_1 = \{0, \frac{1}{2}\}$ and $F_2 = \{\frac{1}{2}, 1\}$;
at each step we halve the distance between two successive node by adding equispaced point with associated Birkhoff data as shown in Figure 6.

Table 1 lists the number of nodes and subsets, as well as the interval width $h$ for the ten set of nodes. In this case $C_{N_{\min}} = 2$, $C_{N_{\max}} = 2$ and $C_{F_{\max}} = 2$.

Figure 7 clearly demonstrate the quadratic approximation order of the operator $M_4 [f_i, \mathcal{F}] (x)$.

For the second experiment, the subsets are generated as follows:

1. we fix 4 equispaced points on the interval $[0, 1]$ and we associate to them the Birkhoff data $f(0), f'(\frac{1}{3}), f''(\frac{1}{3}), f(\frac{2}{3}), f'(1), f''(1)$. In this case we consider the partition of $X$ composed by the subsets $F_1 = \{0, \frac{1}{3}\}$ and $F_2 = \{\frac{2}{3}, 1\}$.

2. at each step we divide by three the distance between two successive node by adding equispaced point with associated Birkhoff data as shown in Figure 8.

Table 2 lists the number of nodes and subsets, as well as the interval width $h$ for the ten set of nodes. In this case $C_{N_{\min}} = 3$, $C_{N_{\max}} = 3$ and $C_{F_{\max}} = 2$. 

Figure 6: Three of the 10 coverings $\mathcal{F}$ used in our numerical experiments with $n = 3$ and $\sharp(\mathcal{F}) = 2$ (upper), $n = 5$ and $\sharp(\mathcal{F}) = 4$ (center), $n = 9$ and $\sharp(\mathcal{F}) = 6$ (down).
Table 1: Starting from $n = 3$ equispaced nodes on the unit interval, we generate coverings $\mathcal{F}$ with $n$ nodes, $m$ subsets $F_k$, and interval width $h$ (compare Figure 6).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1/2</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1/4</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>1/8</td>
</tr>
<tr>
<td>17</td>
<td>16</td>
<td>1/16</td>
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<tr>
<td>33</td>
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<tr>
<td>65</td>
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<td>1/512</td>
</tr>
<tr>
<td>1025</td>
<td>1024</td>
<td>1/1024</td>
</tr>
</tbody>
</table>

Figure 9 clearly demonstrate the cubic approximation order of the operator $M_4 [f, \mathcal{F}] (x)$.

6.2. Approximation accuracy

To test the effectiveness of the Multinode rational interpolation operators we compare them with the corresponding Combined Shepard operators. The numerical results are obtained by locally consider the famous cases of Hermite osculatory, Lidstone and Abel-Goncharov interpolation conditions on $n$ nodes. We solve the local problems by cubic interpolation polynomials on two point with intersecting and subsequent subsets $F_k$. In Tables 3 and 4 we denote by:

1. $S_4^H$ the Shepard-Hermite operator and $M_4^H$ the multinode operator combined with Hermite polynomials of degree 3;

2. $S_4^L$ the Shepard-Lidstone operator and $M_4^L$ the multinode operator combined with Lidstone polynomials of degree 3;

We applied all six operators to the 6 test functions in Figure 8 using a grid of $n$ equispaced points in $[0, 1]$. For $M_4 [f, \mathcal{F}]$ we considered both intersecting $F_k$
Figure 7: Log-log-plot of the approximation error $e_{\text{max}}$ over the interval width for the 6 test functions in Figure 5. As reference, the dotted line indicates a perfect quadratic trend.

Figure 8: Three of the 10 coverings $F$ used in our numerical experiments with $n = 4$ and $\sharp(F) = 2$ (upper), $n = 10$ and $\sharp(F) = 5$ (center), $n = 28$ and $\sharp(F) = 14$ (down).

27
Table 2: Starting from \( n = 4 \) equispaced nodes on the unit interval, we generate coverings \( \mathcal{F} \) with \( n \) nodes, \( m \) subsets \( F_k \), and interval width \( h \) (compare Figure 8).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1/3</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
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<td>14</td>
<td>1/27</td>
</tr>
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<td>82</td>
<td>41</td>
<td>1/81</td>
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<tr>
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<td>1/243</td>
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<tr>
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<td>19684</td>
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<td>1/19683</td>
</tr>
<tr>
<td>59050</td>
<td>29525</td>
<td>1/59049</td>
</tr>
</tbody>
</table>

and disjoint \( F_k \). Tables 3 and 4 lists the maximum error \( e_{\text{max}} \), the average error \( e_{\text{mean}} \) and the mean square error \( e_{\text{MS}} \). The pointwise errors \( e_i \) were determined in absolute value at \( n_e = 1001 \) points of \([0, 1]\) and the errors were calculated by the formulas

\[
e_{\text{max}} = \max_{1 \leq i \leq n_e} e_i, \quad e_{\text{mean}} = \frac{1}{n_e} \sum_{i=1}^{n_e} e_i, \quad e_{\text{MS}} = \sqrt{\frac{\sum_{i=1}^{n_e} e_i^2}{n_e}}.
\]

The results show that the multinode global operator \( M_{\mu} [f, \mathcal{F}] \) is comparable to the Shepard interpolation methods.

7. Conclusions

In this paper we propose to split up an univariate unsolvable Hermite-Birkhoff interpolation problem in two or more solvable subproblems and to blend together the local solutions by using multinode basis functions\(^4\) as blending functions. Numerical experiments are provided which show a good accuracy of approximation and confirm the theoretical results on the approximation order discussed in the paper. It would be of interest to extend this...
Table 3: Comparison of the interpolation operators applied to the 6 test functions in Figure 5 using 33 equispaced interpolation nodes in $[0, 1]$ in the case of Hermite-type data.

<table>
<thead>
<tr>
<th></th>
<th>$S_4^H$</th>
<th>$M_4^H$</th>
<th>$M_1^H$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\epsilon_{\text{max}}$</td>
<td>$\epsilon_{\text{mean}}$</td>
<td>$\epsilon_{\text{MS}}$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>8.9291e-05</td>
<td>1.6655e-05</td>
<td>2.8505e-05</td>
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<tr>
<td>$f_3$</td>
<td>0.00011463</td>
<td>1.3466e-05</td>
<td>2.7513e-05</td>
</tr>
<tr>
<td>$f_6$</td>
<td>0.00016588</td>
<td>1.3466e-05</td>
<td>2.7513e-05</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the interpolation operators applied to the 6 test functions in Figure 5 using 33 equispaced interpolation nodes in $[0, 1]$ in the case of Hermite-type data.
<table>
<thead>
<tr>
<th></th>
<th>$S^L_4$</th>
<th>$M^L_4$ intersecting $F_k$</th>
<th>$F$ partition</th>
</tr>
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<tbody>
<tr>
<td>$e_{\text{max}}$</td>
<td>0.00011326</td>
<td>0.00016864</td>
<td>0.00015177</td>
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<tr>
<td>$f_1$</td>
<td>$e_{\text{mean}}$</td>
<td>2.7617e-05</td>
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<td></td>
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<td>4.702e-05</td>
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<tr>
<td>$e_{\text{max}}$</td>
<td>3.3075e-06</td>
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<td>$f_2$</td>
<td>$e_{\text{mean}}$</td>
<td>7.6442e-07</td>
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<tr>
<td>$e_{\text{max}}$</td>
<td>0.00013568</td>
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<td>$f_3$</td>
<td>$e_{\text{mean}}$</td>
<td>1.428e-05</td>
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<td>$e_{\text{MS}}$</td>
<td>3.1622e-05</td>
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<td>$e_{\text{max}}$</td>
<td>5.7472e-07</td>
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<tr>
<td>$f_4$</td>
<td>$e_{\text{mean}}$</td>
<td>1.4878e-07</td>
<td>8.0795e-08</td>
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<tr>
<td></td>
<td>$e_{\text{MS}}$</td>
<td>2.07e-07</td>
<td>1.2184e-07</td>
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<tr>
<td>$e_{\text{max}}$</td>
<td>1.5514e-06</td>
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<tr>
<td>$f_5$</td>
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<td></td>
<td>$e_{\text{MS}}$</td>
<td>5.5007e-06</td>
<td>6.4193e-06</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the interpolation operators applied to the 6 test functions in Figure 5 using 33 equispaced interpolation nodes in [0, 1] in the case of Lidstone-type data.
Figure 9: Log-log-plot of the approximation error $\varepsilon_{\text{max}}$ over the interval width for the 6 test functions in Figure 5. As reference, the dotted line indicates a perfect cubic trend.

approach to the case of $\mathbb{R}^2$, the sphere $S^2$ and other manifolds, taking into account the results of previously published papers on this topics (See for example [18, 19, 20, 21, 22, 23, 24, 25, 26] and the references therein).

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