Enhancing the approximation order of local Shepard operators by Hermite polynomials

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Abstract

We show how to combine local Shepard operators with Hermite polynomials on the simplex (Chui and Lai, 1990 [3]) so as to raise the algebraic precision of the Shepard-Taylor operators (Farwig, 1986 [8]) that use the same data and contemporaneously maintain the interpolation properties at each sample point (derivative data included) and a good accuracy of approximation. Numerical results are provided.

Key words: Shepard operator, Hermite interpolation, algebraic precision, error analysis

Introduction

Recently a series of papers [12,2,10,11,18,13,21] have focused on the study of a general procedure to increase the algebraic precision, and then the approximation order, of univariate and multivariate approximation schemes using supplementary derivative data. This procedure has been initially proposed in 2003 by X. Han [12] in the univariate case and applies to all schemes that

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reproduce polynomials up to a certain degree, including most common types of interpolation, quasi-interpolation and moving least squares. For an approximation scheme of order $m$ and a dataset that provides $r$ supplementary orders of derivative data at each sample point, the procedure results in an approximation scheme of order $m+r$. Similarly to the well-known procedure that replaces, in the initial approximation operator, each functional evaluation $f(x_i)$ with the Taylor polynomial for $f$ of order $r$ at $x_i$ and results in a new approximation operator of order $\max\{m, r\}$, the new procedure replaces each functional evaluation with a modified (or reduced) version of the Taylor polynomial for $f$ of order $r$. The modified Taylor polynomial is obtained from classic Taylor polynomial by opportuneely reducing its numerical coefficients [12,13]. As one expects, this polynomial does not interpolate the derivative data of any order greater than zero. As a consequence, in the well known case of Shepard or local Shepard operators [17] the use of modified Taylor polynomial [21] instead of classic Taylor polynomial [8] does not increase the precision of the combination, because the degree of exactness of Shepard operators is $0$; moreover, it involves a substantial loss of information, due to the lack of interpolation of supplementary derivative data. In this paper, by following a well established procedure [7,4,6], we show how to combine a local version of Shepard operators with three point interpolation polynomials, so as to raise the algebraic precision of Shepard-Taylor operators [8] that use the same data and contemporaneously maintain the interpolation properties at each sample point (derivative data included) and a good accuracy of approximation. This procedure is based on the association of each sample point with a triangle with a vertex in it and other ones in its neighborhood. The association is realized to minimize the error of the three point interpolation polynomial. To get the interpolation condition of Shepard-Taylor operators and to enhance, at the same time, its algebraic precision, we use specially selected three point Hermite interpolation polynomials; such polynomials are particular cases of a general class of Hermite interpolation polynomials on the simplex well known in literature [3].

The paper is organized as follows. In Section 1 we first present the general problem of the enhancement of the algebraic precision of a linear operator by using supplementary derivative data and then specialize to the particular case of Shepard operator. In section 2 we establish some special notations and terminologies needed to recall the general formulation of Hermite interpolation polynomials on the two dimensional simplex. We work out examples for $r = 1, 2$ that will be used later in the paper. We also provide some useful results concerning error of approximation and limit behaviour of this special kind of polynomial expansions. In Section 3 we define the Shepard-Hermite operators in the bivariate case and study their remainder terms. In Section 4 we provide numerical results obtained by applying the Shepard-Hermite operators to the scattered data interpolation problem in some special situations.
1 Preliminary results and Position of the problem

Let $\Omega \subset \mathbb{R}^s, s \in \mathbb{N}$ be a compact convex domain whose interior is non-empty. Let $\mathcal{N} = \{x^1, x^2, \ldots, x^N\} \subset \Omega, N \in \mathbb{N}$, be a set of $N$ distinct points (called nodes or sample points) with associated functional evaluations, $f(x^1), \ldots, f(x^N)$, $f \in C^r(\Omega)$ and $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_N\} \subset C^r(\Omega)$ be a set of functions which depend only on the node set $\mathcal{N}$. We suppose that functions in $\mathcal{F}$ are cardinal, i.e.

$$\phi_i(x^k) = \delta_{ik} \quad i, k = 1, \ldots, N$$

and form a partition of unity, i.e.

$$\sum_{i=1}^{N} \phi_i(x) = 1.$$  \hfill (2)

Therefore the linear operator

$$L[f](x) = \sum_{i=1}^{N} \phi_i(x) f(x^i)$$ \hfill (3)

interpolates $f$ at each sample point $x^i$ and reproduces exactly constant functions. Let $m \geq 0$ be the algebraic degree of exactness of $L[f]$; of course $m$ will depend on $\mathcal{F}$. In the following we suppose that functional evaluations and all derivatives up to a fixed order $r \geq 1$ are given at each sample point. Under this assumption it is well known that, by replacing functional evaluations $f(x^i)$ in (3) with the Taylor polynomial of order $r$ for $f$ centered at $x^i$, $i = 1, \ldots, N$

$$T_r[f, x^i](x) = \sum_{j=0}^{r} \frac{f^{(j)}(x^i)}{j!} (x - x^i)^j$$ \hfill (4)

we obtain operator

$$L_{T_r}[f](x) = \sum_{i=1}^{N} \phi_i(x) T_r[f, x^i](x)$$ \hfill (5)

with algebraic degree of exactness $\max\{m, r\}$. The idea of Han in the univariate case [12], in embrionic stage in the Ph.D. thesis of Kraaijpoel [13] and subsequently studied by many authors in the multivariate case [10,11,21], consists in the substitution of the Taylor polynomial of order $r$ for $f$ centered at $x^i$ with a modified version $\tilde{T}_r$ of this polynomial defined as follows

$$\tilde{T}_r[f, x^i](x) = \sum_{j=0}^{r} \tilde{a}_j \frac{f^{(j)}(x^i)}{j!} (x - x^i)^j$$ \hfill (6)
with
\[ a_j := \frac{r!(m + r - j)!}{(m + r)!(r - j)!}, \quad j = 0, 1, \ldots, r. \] (7)
It is easy to prove that coefficients \( a_j \) in (7) are all less than 1. For this reason the modified Taylor polynomial is also called the reduced Taylor polynomial [13] and in the following we will use this last appellation. By this modification, we obtain the operator
\[ L_{\tilde{T}_r} [f](x) = \sum_{i=1}^{N} \phi_i(x) \tilde{T}_r[f, \mathbf{x}^i](x) \] (8)
which interpolates \( f \) at each sample point \( \mathbf{x}^i \) and has algebraic degree of exactness \( m + r \).

In the following we will specialize to the case of local Shepard basis functions [17], i.e. we let \( \phi_i(x) = \tilde{W}_{\mu,i}(x) \) with
\[ \tilde{W}_{\mu,i}(x) := \frac{W_{\mu,i}(x)}{\sum_{k=1}^{N} W_{\mu,k}(x)}, \quad \mu > 0 \]
where
\[ W_{\mu,i}(x) := \left( \frac{1}{d_i(x)} - \frac{1}{R_{w_i}} \right)^{\mu}, \]
\( R_{w_i} \) is the radius of influence about node \( \mathbf{x}^i \) and \( d_i(x) \) is the Euclidean distance between \( x \) and \( \mathbf{x}^i \). In practice the radius \( R_{w_i} \) is computed to include at least \( N_w \) points in the open ball \( B(x^i, R_{w_i}) = \{ x : d_i(x) < R_{w_i} \} \) [16]; as a consequence, the value at a point \( x \in \Omega \) of the local Shepard operator
\[ S[f](x) = \sum_{i=1}^{N} \tilde{W}_{\mu,i}(x) f(\mathbf{x}^i) \] (9)
depends only on the data \( N_x = \{ \mathbf{x}^i \in \mathcal{N} : d_i(x) < R_{w_i} \} \). The operator \( S[f] \) has algebraic degree of exactness 0; for \( \mu \in \mathbb{N} \), \( \mu \geq 1 \), basis functions \( \tilde{W}_{\mu,i}(x) \) are at least of class \( C^{\mu-1}(\Omega) \) and all partial derivatives of \( \tilde{W}_{\mu,i} \) up to the order \( \mu - 1 \) are zero:
\[ \tilde{W}_{\mu,i}^{(p_1, \ldots, p_s)}(x^k) = 0, \quad k = 1, \ldots, N; (p_1, \ldots, p_s) \in \mathbb{Z}_+^s, 1 \leq p_1 + \ldots + p_s \leq \mu - 1; \]
as a consequence the local Shepard-Taylor operator [8]
\[ S_{T_r}[f](x) = \sum_{i=1}^{N} \tilde{W}_{\mu,i}(x) \sum_{j=0}^{r} \frac{f^{(j)}(\mathbf{x}^i)}{j!} (x - \mathbf{x}^i) \] (10)
has algebraic degree of exactness \( r \) and interpolates on all data required for its definition, provided that \( \mu > r + 1 \).
By replacing the Taylor polynomial $T_r[f]$ with the reduced Taylor polynomial $\widetilde{T}_r[f]$ we obtain the Shepard-reduced Taylor operator recently studied in [21]

$$S_{\widetilde{T}_r}[f](x) = \sum_{i=1}^{N} \tilde{W}_{\mu,i}(x) \sum_{j=0}^{r} a_j \frac{f^{(j)}(x^i)}{j!} (x - x^i). \quad (11)$$

This replacement does not increase the degree of exactness of the combined operator since the local Shepard operator has degree of exactness $m = 0$. Moreover the resulting operator (11) loses interpolation properties of the Shepard-Taylor operator (10) which uses the same data, with the exception of the interpolation of functional evaluations: in fact the reduced Taylor polynomial (6) does not interpolate successive derivatives of $f$ at the node $x^i$ of any order greater than zero. Then a question arises. By assuming that all functional evaluations and supplementary derivative data up to a fixed order $r$ are given at each node $x^i$, we wonder if it is possible to enhance the degree of exactness of the Shepard operator to $p = r + q$, with $q > 0$, maintaining the interpolation properties of the Shepard-Taylor operator $S_{T_r}[f]$ and the accuracy of approximation of the operator $S_{T_p}[f]$. In this paper we give a positive answer to the previous question by combining in a special way local Shepard operators with well known Hermite interpolation polynomials on the simplex [3]. To simplify notations we will consider only the bivariate case, but the procedure we propose has a direct generalization to the $s$-dimensional case.

2 Some remarks on the Hermite polynomial on the simplex

The formulation of the Hermite interpolation polynomials on the vertices of a simplex $\Delta_n \subset \mathbb{R}^n$, given by Chui and Lai in a famous paper of 1991 [3] in connection with the notion of super vertex splines, is here readapted to $\Delta_2 \subset \mathbb{R}^2$ for our purpose. It is very technical and requires some preliminary notations and definitions.

Let us denote by $\mathbb{Z}^2_+$ the set of all couples with non-negative integer components in the euclidean space $\mathbb{R}^2$. As usual, for $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2_+$, we will use the notations $|\beta| = \beta_1 + \beta_2$, $\beta! = \beta_1! \beta_2!$ and $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for all $i = 1, 2$. Let $x^i = (x^i_1, x^i_2)$, $i = 0, 1, 2$ be not collinear points in an anticlockwise ordering and let denote by $\Delta_2$ the two-dimensional simplex of vertices $x^0, x^1, x^2$. The barycentric coordinates $(\lambda_0(x), \lambda_1(x), \lambda_2(x))$ of a generic point $x \in \mathbb{R}^2$ relative to the simplex $\Delta_2$, are defined by

$$\lambda_0(x) = \frac{A(x, x^1, x^2)}{A(x^0, x^1, x^2)}, \quad \lambda_1(x) = \frac{A(x^0, x, x^2)}{A(x^0, x^1, x^2)}, \quad \lambda_2(x) = \frac{A(x^0, x^1, x)}{A(x^0, x^1, x^2)}$$

where $A(x^0, x^1, x^2)$ is the signed area of the simplex of vertices $x^0, x^1, x^2$. We
will use also notations
\[ D_{ij} f = (x^i - x^j) \cdot \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) , \quad i, j = 0, 1, 2 \]
for the derivative of \( f \) along the directed line segment from \( x^j \) to \( x^i \) and
\[ D^\beta_0 := D_{10}^\beta D_{20}^\beta, \quad D_1^\beta := D_{01}^\beta D_{21}^\beta, \quad D_2^\beta := D_{02}^\beta D_{12}^\beta \]
for the composition of derivatives along the directed sides of the simplex. In addition, let \( c_i, i = 0, 1, 2 \) be the projection from \( \mathbb{Z}_3^+ \) to \( \mathbb{Z}_2^+ \) which associates to each \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_3^+ \) the couple obtained from \( \alpha \) deprived of the component \( i \) respectively, i.e.
\[ c_0 \alpha = (\alpha_1, \alpha_2), \quad c_1 \alpha = (\alpha_0, \alpha_2), \quad c_2 \alpha = (\alpha_0, \alpha_1). \]
The following Theorem was proven in [3, Theorem 3.1.1] in the general case of \( \Delta_n \subset \mathbb{R}^n \).

**Theorem 1** In Bézier representation with respect to \( \Delta_2 \), the Taylor polynomial of order \( p \) of a sufficiently smooth function \( f \) centered at the vertex \( x^0 \) is given by
\[
T_p \left[ f, x^0 \right] (x) = \sum_{|\alpha| = p} \sum_{\alpha \in \mathbb{Z}_3^+, \beta \leq \alpha} \left( \frac{c_0 \alpha}{\beta} \right) (p - |\beta|)! \frac{D_0^\beta f \left( x^0 \right) \phi^p \left( \lambda_0 (x), \lambda_1 (x), \lambda_2 (x) \right)}{p!} \]
where
\[
\phi^p \left( \lambda_0 (x), \lambda_1 (x), \lambda_2 (x) \right) = \frac{p!}{\alpha!} \lambda_0 (x)^{\alpha_0} \lambda_1 (x)^{\alpha_1} \lambda_2 (x)^{\alpha_2}.
\]

Now we will specify certain Hermite-type interpolation conditions on the vertices of a two–dimensional simplex which ensure uniqueness of interpolation in bivariate polynomial spaces \( P^p_x \) of total degree not greater than \( p \). For this purpose, we need some additional definitions.

**Definition 2** A subset \( M^2 \) of \( \mathbb{Z}_3^2 \) is called a lower set if for each \( \beta, \gamma \in \mathbb{Z}_3^2, \beta \in M^2 \) and \( 0 \leq \gamma, \beta \) it results \( \gamma \in M^2 \).

Let \( \Gamma^2_r := \{ \beta \in \mathbb{Z}_3^2 : |\beta| \leq r \} \), \( \Lambda^3_r := \{ \alpha \in \mathbb{Z}_3^3 : |\alpha| = r \} \) and \( A^r_\gamma \) the raising map from \( \Gamma^2_r \) to \( \Lambda^3_r \) defined by
\[ A^r_0 \beta = (r - |\beta|, \beta_1, \beta_2), \quad A^r_1 \beta = (\beta_1, r - |\beta|, \beta_2), \quad A^r_2 \beta = (\beta_1, \beta_2, r - |\beta|), \beta \in \mathbb{Z}_3^2. \]

**Definition 3** A collection of subsets \( M^2_0, M^2_1, M^2_2 \) of \( \Gamma^2_r \) is said to form a partition of \( \Lambda^3_r \) if
The following Theorem was proven in [3, Theorem 3.1.4] in the general case of $\Delta_n \subset \mathbb{R}^n$ and lower sets $M^n_0, M^n_1, ..., M^n_n$ forming a partition of $\Lambda^{n+1}_r$.

**Theorem 4** Let $M^2_0 = \{ \beta \in \mathbb{Z}_+^2 : |\beta| \leq r \}$ and $M^2_2$, $M^2_2$ lower sets forming a partition of $\Lambda^2_r$. Then for any given set of data $\{ f_{i, \beta} \in \mathbb{R} : \beta \in M^2_i, i = 0, 1, 2 \}$ there exists a unique polynomial $H^\Delta_{p,r}$ of total degree $p = r + q$ satisfying

$$D^i_\beta H^\Delta_{p,r} (x^i) = f_{i, \beta}, \: \beta \in M^2_i, \: i = 0, 1, 2.$$

Moreover, $H^\Delta_{p,r} (x)$ may be formulated in the Bézier representation of total degree $p$ with respect to the simplex $\Delta_2$ as follows

$$H^\Delta_{p,r} (x) = \sum_{i=0}^{p+q} \sum_{\beta \in M^2_i} \left( \sum_{\gamma \leq \beta} \binom{p+q}{\gamma} \frac{(p-|\gamma|)!}{p!} f_{i, \gamma} \right) \phi_{A^p_i \beta}^i (\lambda_0 (x), \lambda_1 (x), \lambda_2 (x))$$

(13)

where

$$\phi_{A^p_i \beta}^i (\lambda_0 (x), \lambda_1 (x), \lambda_2 (x)) = \frac{p!}{(A^p_i \beta)!} \lambda_0 (x)^{(A^p_i \beta)_0} \lambda_1 (x)^{(A^p_i \beta)_1} \lambda_2 (x)^{(A^p_i \beta)_2}.$$

Let $f$ be a function of class $C^p (\Omega)$. Following above notations we set $f_{i, \beta} = D^i_\beta f (x^i) \beta \in M^2_i, i = 0, 1, 2$ and $H^\Delta_{p,r}[f] (x) := H^\Delta_{p,r} (x)$; then for each $x \in \Omega$ we have

$$f (x) = H^\Delta_{p,r}[f] (x) + R^H_{p,r, \Delta_2} [f] (x)$$

(14)

where $R^H_{p,r, \Delta_2} [f] (x)$ is the remainder term. We emphasize that $H^\Delta_{p,r}[f]$ interpolates functional evaluations of $f$ at the vertices, all partial derivatives of $f$ up to the order $r$ at $x^0$ and some directional derivatives at the remaining vertices for a total of $\dim (P^p_\Omega) \binom{p+2}{2}$ interpolation conditions. As a result of the uniqueness of interpolation, $H^\Delta_{p,r}[:]$ leaves invariant all polynomials of total degree not greater than $p$. The last property is particularly useful since it allows us to get explicit expression for the remainder $R^{H^\Delta_{p,r}}_{p,r, \Delta_2} [f] (x)$. In fact, let us assume that $f \in C^p (\Omega)$ and that the partial derivatives $\frac{\partial^p f}{\partial x^{p-i} \partial y^i}$ of order $p$ are Lipschitz continuous in $\Omega$ for each $i = 0, \ldots, p$. We denote the space of such functions with $C^{p,1} (\Omega)$ and we associate to this space the seminorm

$$|f|_{p,1} = \sup_{i=0, \ldots, p} \left\{ \frac{|\frac{\partial^p f}{\partial x^{p-i} \partial y^i} (x^1) - \frac{\partial^p f}{\partial x^{p-i} \partial y^i} (x^2)|}{\|x^1 - x^2\|_2}, \: x^1 \neq x^2 \in \Omega \right\}.$$
\(2 \times A(\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2)\). By the Cauchy-Schwartz inequality it follows that

\[
|\lambda_i(\mathbf{x})| \leq \rho S \|\mathbf{x} - \mathbf{x}^0\|_2, \quad i = 1, 2.
\]  

(15)

Moreover, by the binomial theorem it results

\[
\lambda_0^k(\mathbf{x}) = (1 - \lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x}))^k
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \lambda_1^{i-j}(\mathbf{x}) \lambda_2^j(\mathbf{x})
\]

and then

\[
|\lambda_0^k(\mathbf{x})| \leq \left(\rho S \|\mathbf{x} - \mathbf{x}^0\|_2\right)^k \sum_{i=0}^{k} \binom{k}{i} \sum_{j=0}^{i} \binom{i}{j}.
\]

(16)

The following Theorem holds.

**Theorem 5** Let \(\Omega\) be a compact convex domain containing \(\Delta_2\) and \(f \in C^{p,1}(\Omega)\). Then, for each \(\mathbf{x} \in \Omega\), we have

\[
H_{p,r}^{\Delta_2}[f](\mathbf{x}) = T_p[f, \mathbf{x}^0](\mathbf{x}) + \delta_{p,r}^{\Delta_2}(\mathbf{x})
\]

(17)

where \(T_p[f, \mathbf{x}^0](\mathbf{x})\) is the Taylor polynomial of order \(p\) for \(f\) centered at \(\mathbf{x}^0\) and

\[
|D_{01}^{\kappa_1} D_{02}^{\kappa_2} \delta_{p,r}^{\Delta_2}(\mathbf{x})| \leq |f|_{p,1} \rho^{2+|\kappa|} \left(\rho S\right)^{p-|\kappa|} \max \left\{ \rho^{p-|\kappa|}, \|\mathbf{x} - \mathbf{x}^0\|_2^{p-|\kappa|} \right\} K_\kappa(p)
\]

(18)

for each \(\kappa = (\kappa_1, \kappa_2)\) s.t. \(|\kappa| \leq p\). \(K_\kappa(p)\) are explicitly computable constants independent on \(f\) or \(\mathbf{x}\).

**PROOF.** For \(\kappa = (0,0)\) we rewrite (13) by expanding the sum of index \(i\), i.e.

\[
H_{p,r}^{\Delta_2}(\mathbf{x}) = \sum_{\beta \in M_0^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(p-|\gamma|)!}{p!} D_{0}^{\gamma} f(\mathbf{x}^0) \right\} \phi_{A_0}^{\beta}(\lambda_0(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))
\]

\[
+ \sum_{\beta \in M_2^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(p-|\gamma|)!}{p!} D_{1}^{\gamma} f(\mathbf{x}^1) \right\} \phi_{A_1}^{\beta}(\lambda_0(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))
\]

\[
+ \sum_{\beta \in M_2^2} \left\{ \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(p-|\gamma|)!}{p!} D_{2}^{\gamma} f(\mathbf{x}^2) \right\} \phi_{A_2}^{\beta}(\lambda_0(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})).
\]

(19)
By the Taylor Theorem with integral remainder we get

\[
T_{p-|\gamma|} [D_1^\gamma f, x^0] (x^1) = \sum_{|\alpha|=p-|\gamma|} \sum_{\delta \leq \alpha \in \mathbb{Z}_+^3, \delta \in \mathbb{Z}_+^2} \frac{(p-|\delta|-|\gamma|)!}{\delta_1! \delta_2! (\alpha_1 - \delta_1)! (\alpha_2 - \delta_2)!} \\
\times \frac{D_0^\delta D_1 f (x^0) \lambda_0 (x^1)^{\alpha_0} \lambda_1 (x^1)^{\alpha_1} \lambda_2 (x^1)^{\alpha_2}}{p-|\gamma|+1} \\
+ \int_0^1 \frac{D_0^{p-\gamma+1} D_1^\gamma f (x^0 + t (x^1 - x^0))}{(p-|\gamma|+1)!} (1-t)^{p-|\gamma|+1} dt
\]

and

\[
T_{p-|\gamma|} [D_2^\gamma f, x^0] (x^2) = \sum_{|\alpha|=p-|\gamma|} \sum_{\delta \leq \alpha \in \mathbb{Z}_+^3, \delta \in \mathbb{Z}_+^2} \frac{(p-|\delta|-|\gamma|)!}{\delta_1! \delta_2! (\alpha_1 - \delta_1)! (\alpha_2 - \delta_2)!} \\
\times \frac{D_0^\delta D_2 f (x^0) \lambda_0 (x^2)^{\alpha_0} \lambda_1 (x^2)^{\alpha_1} \lambda_2 (x^2)^{\alpha_2}}{p-|\gamma|+1} \\
+ \int_0^1 \frac{D_0^{p-\gamma+1} D_2^\gamma f (x^0 + t (x^2 - x^0))}{(p-|\gamma|+1)!} (1-t)^{p-|\gamma|+1} dt
\]

where \( p - \gamma + 1 := (p - \gamma_1 + 1, p - \gamma_2 + 1) \). By the property

\[
\lambda_i (x^j) = \delta_{ij}, \quad i, j = 0, 1, 2
\]

it follows that

\[
T_{p-|\gamma|} [D_1^\gamma f, x^0] (x^1) = \frac{1}{\delta_1!} D_1^{\delta_1} D_1^\gamma f (x^0) \\
+ \int_0^1 \frac{D_0^{p-\gamma+1} D_1^\gamma f (x^0 + t (x^1 - x^0))}{p-|\gamma|+1} (1-t)^{p-|\gamma|+1} dt
\]

and

\[
T_{p-|\gamma|} [D_2^\gamma f, x^0] (x^2) = \frac{1}{\delta_2!} D_2^{\delta_2} D_2^\gamma f (x^0) \\
+ \int_0^1 \frac{D_0^{p-\gamma+1} D_2^\gamma f (x^0 + t (x^2 - x^0))}{p-|\gamma|+1} (1-t)^{p-|\gamma|+1} dt.
\]

By substituting the relations (20) and (21) in (19) we have

\[
H_{p,r}^\Delta (x) = T_p [f, x^0] (x) + \delta_{p,r}^\Delta (x)
\]
where

$$
\delta_{p,r}^2 (x) = \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{p!} \int_0^1 D_{\alpha}^{p-|\gamma|+1} f (x^0 + t (x^1 - x^0)) \frac{1}{(p-|\gamma|+1)!} (1-t)^{p-|\gamma|+1} dt
$$

\begin{align*}
&\times \frac{1}{(p-|\beta|)! \alpha_1! \alpha_2!} \lambda_0 (x)^{\alpha_1} \lambda_1 (x)^{p-|\beta|} \lambda_2 (x)^{\alpha_2} \\
&+ \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{p!} \int_0^1 D_{\alpha_1}^{p-|\gamma|+1} D_{\alpha_2}^{p-|\beta|} f (x^0 + t (x^2 - x^0)) \frac{1}{(n-|\gamma|+1)!} (1-t)^{p-|\gamma|+1} dt \\
&\times \frac{1}{(p-|\beta|)! \alpha_1! \alpha_2!} \lambda_0 (x)^{\alpha_1} \lambda_1 (x)^{p-|\beta|} \lambda_2 (x)^{\alpha_2} \\
&+ \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{p!} \sum_{\delta \leq p-|\gamma|} \frac{D_{\alpha_1}^{p-|\beta|} f (x^0)}{\delta_1!} \phi_{A_1,\beta}^{p+1} (x^0, \lambda_0 (x), \lambda_1 (x), \lambda_2 (x))
\end{align*}

and

$$
T_p \left[ f, x^0 \right] (x) = \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{p!} D_{\alpha}^{p+1} f (x^0) \phi_{A_1,\beta}^{p+1} (x^0, \lambda_0 (x), \lambda_1 (x), \lambda_2 (x))
$$

\begin{align*}
&+ \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{p!} \sum_{\delta \leq p-|\gamma|} \frac{D_{\alpha_1}^{p+1} f (x^0)}{\delta_1!} \phi_{A_1,\beta}^{p+1} (x^0, \lambda_0 (x), \lambda_1 (x), \lambda_2 (x))
\end{align*}

To obtain the bound (18) in the considered case we take the modulus of both sides of (22); by using relations (15) and (16) we have

$$
\frac{|\delta_{p,r}^2 (x)|}{|f|_{p,1}} \leq \rho^{p+2} (\rho^2 S)^p \left( \frac{\|x - x^0\|_2}{\rho} \right)^p \left( \sum_{\beta \in M_2} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{(p-|\gamma|+2)!} (p-|\beta|)! \beta_1! \beta_2! \right)
$$

The result follows by setting

$$
K_{(0,0)} (p) = \sum_{\beta \in M_2} \left\{ \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \frac{(p-|\gamma|)!}{(p-|\gamma|+2)!} (p-|\beta|)! \beta_1! \beta_2! \right\}
$$

and by distinguishing the cases $\|x - x^0\|_2 \leq \rho$ and $\|x - x^0\|_2 > \rho$. The cases $|\kappa| \geq 1$ can be obtained analogously by considering the derivative of $\delta_{p,r}^2 (x)$. It remains to prove that $T_p \left[ f, x^0 \right] (x)$ is the Taylor polynomial for $f$ of order $p$ centered at $x^0$ (12). For this purpose we note that from the bounds (18) we
\[ \lim_{\rho \to 0} \left\| \delta_{p,r}^2 (x) \right\|_{W^{p,\infty}(\Delta_2)} = 0 \]

where we have denoted with \( \left\| \cdot \right\|_{W^{p,\infty}(\Delta_2)} \) the Sobolev norm

\[ \left\| f \right\|_{W^{p,\infty}(\Delta_2)} = \sum_{|\kappa| \leq p} \left\| D_{01}^{\kappa_1} D_{02}^{\kappa_2} f \right\|_{L^{\infty}(\Delta_2)} . \]

By setting \( \alpha \) and \( \beta \) as the adjacent angles to the side of length \( \rho \), we see that \( \rho^2 S = \frac{\sin(\alpha+\beta)}{\sin \alpha \sin \beta} \) depends only on the form of simplex \( \Delta_2 \). Therefore, since \( H_{p,r}^2 \) leaves invariant all polynomials of total degree not greater than \( p \) it follows from [5, Theorem 5] that

\[ \lim_{\rho \to 0} \left\| f (x) - H_{p,r}^2 [f] (x) \right\|_{W^{p,\infty}(\Delta_2)} = 0 \]

and therefore by the triangle inequality we get

\[ \left\| f (x) - T_p \left[ f, x^0 \right] (x) \right\|_{W^{p,\infty}(\{x^0\})} \leq \lim_{\rho \to 0} \left\| f (x) - H_{p,r}^2 [f] (x) \right\|_{W^{p,\infty}(\Delta_2)} \]

\[ + \lim_{\rho \to 0} \left\| \delta_{p,r}^2 (x) \right\|_{W^{p,\infty}(\Delta_2)} = 0. \]

Consequently all partial derivatives up to the order \( p \) of \( T_p \left[ f, x^0 \right] (x) \) at \( x^0 \) coincide with corresponding partial derivatives of \( f \) at \( x^0 \). The result follows by the uniqueness of classic Taylor interpolation [1] noting that \( T_p \left[ f, x^0 \right] (x) \) is a polynomial of degree not greater than \( p \).

**Corollary 6** In the hypothesis of Theorem 5 for all \( x \in \Omega \) we have

\[ R_{H,r}^2 [f] (x) = R_p^T \left[ f, x^0 \right] (x) - \delta_{p,r}^2 (x) \] (23)

where \( R_p^T \left[ f, x^0 \right] (x) \) is the remainder term in Taylor expansion; moreover

\[ \left| D_{01}^{\kappa_1} D_{02}^{\kappa_2} R_{H,r}^2 [f] (x) \right| \leq \left| f \right|_{p,1} \rho^{2+|\kappa|} \left( \rho^2 S \right)^{p-|\kappa|} \max \left\{ \rho^{p-|\kappa|}, \left\| x - x^0 \right\|_2^{p-|\kappa|} \right\} \tilde{K}_\kappa (p) \] (24)

with

\[ \tilde{K}_\kappa (p) = \frac{2^{p-|\kappa|}}{(p - |\kappa| - 1)!} + K_\kappa (p). \] (25)

**PROOF.** Equation (23) trivially follows from equation (17); bounds (24) follow from the triangular inequality by bounding Taylor remainder in a classic way [7].

11
2.1 Particular cases

Of particular interest are expansions (14) for \( r = 1, 2 \) that will be used later in the paper. In order to get these expansions, we graphically represent the corresponding lower sets by Bézier nets as in [3]. In particular we denote functional evaluations by balls on the vertices of the simplex and derivatives along the directed line segment by balls on the sides; mixed derivatives are then denoted by balls in the interior of the simplex. As in Theorem 4, we fix \( M^2_0 \) such that we interpolate \( f \) and all its partial derivatives up to the order \( r \) at the vertex \( x^0 \) and lower sets \( M^2_1, M^2_2 \) so as to maintain a certain symmetry in the distribution of the remaining interpolation conditions at the vertices \( x^1, x^2 \).

2.1.1 Case \( r = 1 \)

Let us suppose that functional evaluations and first order derivatives of a function \( f \) are given at the vertices of \( \Delta_2 \), that is \( r = 1 \). Let us fix lower sets

\[
M^2_0 = \{(0,0),(1,0),(0,1)\}, \\
M^2_1 = \{(0,0),(0,1)\}, \\
M^2_2 = \{(0,0)\}. 
\]  

(26)

In Figure 1 we graphically represent the Bézier net according to the corresponding \( M^2_i, i = 0, 1, 2 \) in (26). The corresponding interpolation polynomial...
in Bézier representation is

\[
H_{2,1}^A [f] (x) = f (x^0) \lambda_0 (x) \lambda_0 (x) + 2\lambda_1 (x) + 2\lambda_2 (x)) \\
+ f (x^1) \lambda_1 (x) \lambda_1 (x) + 2\lambda_2 (x)) + f (x^2) \lambda_2^2 (x) + D_{10} f (x^0) \lambda_0 (x) \lambda_1 (x) \\
+ D_{20} f (x^0) \lambda_0 (x) \lambda_2 (x) + D_{21} f (x^1) \lambda_1 (x) \lambda_2 (x).
\] (27)

The polynomial \( H_{2,1}^A [f] (x) \) has algebraic degree of exactness 2. Therefore in the case \( r = 1 \), \( H_{2,1}^A \) has degree of precision increased by 1 with respect to the Taylor polynomial that uses the same data at \( x^0 \).

2.1.2 Case \( r = 2 \)

Let us suppose that functional evaluations, first and second order derivatives of a function \( f \) are given at the vertices of \( \Delta_2 \), that is \( r = 2 \). Let us fix lower sets

\[
M_0^2 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}, \\
M_1^2 = \{(0,0), (0,1)\}, \\
M_2^2 = \{(0,0), (0,1)\}.
\] (28)

In Figure 2 we graphically represent the Bézier net according to the corresponding \( M_i^2 \), \( i = 0, 1, 2 \) in (28). In this case we fix lower sets \( M_0^2, M_1^2 \) to obtain an interpolant with degree of exactness 3, that is, we fix 10 interpolation
The polynomial $H^{\Delta_2}_{3,2} [f] (x)$ has degree of precision 3. Therefore also in the case $r = 2$, $H^{\Delta_2}_{3,2}$ has degree of precision increased by 1 with respect to the Taylor polynomial that uses the same data at $x^0$. Nevertheless, in case $r = 2$ it is possible to fix others lower sets, as for example those depicted in figure 3. In this case we obtain polynomial

$$H^{\Delta_2}_{3,2} [f] (x) = f (x^0) \lambda_0 (x) (\lambda_0 (x) (\lambda_0 (x) + 3\lambda_1 (x) + 3\lambda_2 (x))$$

$$+ 3(\lambda_1 (x) + \lambda_2 (x))^2 + f (x^1) \lambda_1^2 (x) (\lambda_1 (x) + 3\lambda_2 (x))$$

$$+ f (x^2) \lambda_2^2 (x) (3\lambda_1 (x) + \lambda_2 (x))$$

$$+ D_{10} f (x^0) \lambda_0 (x) \lambda_1 (x) (\lambda_0 (x) + 2\lambda_1 (x) + 2\lambda_2 (x))$$

$$+ D_{20} f (x^0) \lambda_0 (x) \lambda_2 (x) (\lambda_0 (x) + 2\lambda_1 (x) + 2\lambda_2 (x))$$

$$+ D_{21} f (x^1) \lambda_1^2 (x) \lambda_2 (x) + D_{12} f (x^2) \lambda_1 (x) \lambda_2^2 (x)$$

$$+ \frac{1}{2} D_{10}^2 f (x^0) \lambda_0 (x) \lambda_1^2 (x) + \frac{1}{2} D_{20}^2 f (x^0) \lambda_0 (x) \lambda_2^2 (x)$$

$$+ D_{10} D_{20} f (x^0) \lambda_0 (x) \lambda_1 (x) \lambda_2 (x)$$

(29)
\[ H_{4,2}^{n_2} [f] (\mathbf{x}) = f (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) (\lambda_0^2 (\mathbf{x}) + 2 (\lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x})) (2 \lambda_0 (\mathbf{x}) + 3 (\lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x}))) \]
\[ + f (\mathbf{x}^1) \lambda_0^2 (\mathbf{x}) (4 \lambda_0 (\mathbf{x}) (\lambda_1 (\mathbf{x}) + 3 \lambda_2 (\mathbf{x})) + \lambda_1 (\mathbf{x}) (\lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x}))) \]
\[ + f (\mathbf{x}^2) \lambda_0^2 (\mathbf{x}) (3 \lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x})) (4 \lambda_0 (\mathbf{x}) + 2 \lambda_1 (\mathbf{x})) + \lambda_2 (\mathbf{x}) (2 \lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x}))) \]
\[ + D_{10} f (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) \lambda_1 (\mathbf{x}) (\lambda_0 (\mathbf{x}) + 3 \lambda_1 (\mathbf{x}) + 3 \lambda_2 (\mathbf{x})) \]
\[ + D_{20} f (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) \lambda_2 (\mathbf{x}) (\lambda_0 (\mathbf{x}) + 3 \lambda_1 (\mathbf{x}) + 3 \lambda_2 (\mathbf{x})) \]
\[ + D_{01} f (\mathbf{x}^1) \lambda_0 (\mathbf{x}) \lambda_1^2 (\mathbf{x}) (\lambda_1 (\mathbf{x}) + 3 \lambda_2 (\mathbf{x})) \]
\[ + D_{21} f (\mathbf{x}^1) \lambda_1^2 (\mathbf{x}) \lambda_2 (\mathbf{x}) (3 \lambda_0 (\mathbf{x}) + \lambda_1 (\mathbf{x})) \]
\[ + D_{02} f (\mathbf{x}^2) \lambda_0 (\mathbf{x}) \lambda_2^2 (\mathbf{x}) (3 \lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x})) \]
\[ + D_{12} f (\mathbf{x}^2) \lambda_1 (\mathbf{x}) \lambda_2^2 (\mathbf{x}) (3 \lambda_0 (\mathbf{x}) + 3 \lambda_1 (\mathbf{x}) + \lambda_2 (\mathbf{x})) \]
\[ + \frac{1}{2} D_{10}^2 (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) \lambda_1^2 (\mathbf{x}) + \frac{1}{2} D_{20} f (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) \lambda_2^2 (\mathbf{x}) \]
\[ + D_{10} D_{20} f (\mathbf{x}^0) \lambda_0^2 (\mathbf{x}) \lambda_1 (\mathbf{x}) \lambda_2 (\mathbf{x}) + D_{01} D_{21} f (\mathbf{x}^1) \lambda_0 (\mathbf{x}) \lambda_1^2 (\mathbf{x}) \lambda_2 (\mathbf{x}) \]
\[ + \frac{1}{2} D_{12}^2 f (\mathbf{x}^2) \lambda_1^2 (\mathbf{x}) \lambda_2 (\mathbf{x})^2 + D_{02} D_{12} f (\mathbf{x}^2) \lambda_0 (\mathbf{x}) \lambda_1 (\mathbf{x}) \lambda_2^2 (\mathbf{x}) . \]

(30)

The degree of precision of \( H_{4,2}^{n_2} \) is 4, hence it increases by 2 the degree of precision of the Taylor polynomial that uses the same data at \( \mathbf{x}^0 \).

3 The bivariate Shepard-Hermite operator

For each \( i = 1, \ldots, N \) we associate to \( \mathbf{x}^i \) the simplex \( \Delta_2 (i) \) with fixed vertex \( \mathbf{x}^i \) and vertices in \( B (\mathbf{x}^i, R_{w_i}) \) (see Sec. 1) which minimize the quantity \( \rho_i^2 (\rho_i^2 S_i)^p \) according to Corollary 6 Theorem 5.

**Definition 7** For each \( \mu > 0 \) and \( p = 1, 2, \ldots \) the bivariate Shepard-Hermite operator is defined by

\[ S_{H_{p,r}} [f] (\mathbf{x}) = \sum_{i=1}^{N} \widetilde{W}_{\mu,i} (\mathbf{x}) H_{p,r}^{\Delta_2(i)} [f] (\mathbf{x}) , \mathbf{x} \in \Omega \]

(31)

where \( H_{p,r}^{\Delta_2(i)} [f] (\mathbf{x}) \) is the Hermite interpolating polynomial (13) on the simplex \( \Delta_2 (i) \), \( i = 1, \ldots, N \). The remainder term is

\[ R_{H_{p,r}} [f] (\mathbf{x}) = f (\mathbf{x}) - S_{H_{p,r}} [f] (\mathbf{x}) , \mathbf{x} \in \Omega . \]

(32)

Convergence results can be obtained by following the well known approaches [19, §15.4], [7,20]. We set:
(1) \( \mathcal{I}_x = \{ i \in \{1, \ldots, N\} : |x - x^i| < R_{w_i}, x \in \Omega \}; \)
(2) \( M = \sup_{x \in \Omega} \{ \text{card} (\mathcal{I}_x) \}; \)
(3) \( d_i = 2R_{w_i}, i = 1, \ldots, N; \)
(4) \( C_{\alpha, \beta} > 0, \alpha, \beta \in \mathbb{N} : 0 < \alpha + \beta \leq \mu - 1 \) constants satisfying
\[
\sup_{x \in B(x^i, R_{w_i})} \left| \frac{\partial^{\alpha+\beta} \tilde{W}_{\mu,i} (x)}{\partial x^{\alpha} \partial y^{\beta}} \right| \leq C_{\alpha, \beta} \frac{d_i^{\alpha+\beta}}{d_i^{\alpha+\beta}};
\]
(5) \( \tau_i = \rho_i^2 S_i, i = 1, \ldots, N. \)

**Theorem 8** We have
\[
\sup_{x \in \Omega} |R_{H_{p,r}} [f] (x)| \leq \tilde{K}_{(0,0)} (p) |f|_{p,1} \max_{i \in \mathcal{I}(x)} (\rho_i^2 \tau_i^p d_i^p)
\]
where \( \tilde{K}_{(0,0)} (p) \) is the constant defined in (25).

**PROOF.** As the Shepard basis form a partition of unity (2), we get
\[
R_{H_{p,r}} [f] (x) = f (x) - S_{H_{p,r}} [f] (x) = \sum_{i \in \mathcal{I}(x)} \tilde{W}_{\mu,i} (x) \left( f (x) - H_{p,r}^{\Delta_2 (i)} [f] (x) \right)
\]
and therefore, from (24) it results
\[
|R_{H_{p,r}} [f] (x)| \leq \sum_{i \in \mathcal{I}(x)} \tilde{W}_{\mu,i} (x) |f (x) - H_{p,r}^{\Delta_2 (i)} [f] (x)| \leq \sum_{i \in \mathcal{I}(x)} \tilde{W}_{\mu,i} (x) |f|_{p,1} \max_{i \in \mathcal{I}(x)} (\rho_i^2 \tau_i^p d_i^p \tilde{K}_{(0,0)} (p))
\]

**Theorem 9** Let \( f \) be a function of class \( C^{p,1} (\Omega) \). Then for each \( \alpha, \beta \in \mathbb{N} \) such that \( 0 < \alpha + \beta < \mu \) the following bound holds:
\[
\sup_{x \in \Omega} \left| \frac{\partial^{\alpha+\beta} R_{H_{p,r}} [f]}{\partial x^{\alpha} \partial y^{\beta}} \right| \leq C_{\alpha, \beta} M \max_{i \in \mathcal{I}_x} \left( \rho_i^{2+\alpha+\beta} \tau_i^p d_i^{p-(\alpha+\beta)} \right) |f|_{p,1}
\]
where
\[
C_{\alpha, \beta} = \sum_{|\kappa| \leq \alpha+\beta} \binom{\alpha}{\kappa_1} \binom{\beta}{\kappa_2} C_{\alpha-\kappa_1, \beta-\kappa_2} \tilde{K}_\kappa (p)
\]
with \( \tilde{K}_\kappa (p) \) as in Theorem 5 and \( C_{\alpha-\kappa_1, \beta-\kappa_2} \) defined above.
**PROOF.** By differentiating $\alpha$ times with respect to $x$ and $\beta$ times with respect to $y$, $0 < \alpha + \beta < \mu$, both sides of (32), by using Leibniz’ rule, we get

$$
\frac{\partial^{\alpha+\beta} R_{H,p,r} [f] (x)}{\partial x^\alpha \partial y^\beta} = \sum_{j \in I} \sum_{|\kappa| \leq \alpha + \beta} \left( \frac{\alpha}{\kappa_1} \right) \left( \frac{\beta}{\kappa_2} \right) \frac{\partial^{\alpha+\beta-|\kappa|} \tilde{W}_{\mu,j} (x, y) \partial^{|\kappa|} R_{H,p,r}^\Delta : f (x)}{\partial x^{\alpha-\kappa_1} \partial y^{\beta-\kappa_2}}.
$$

Therefore

$$
\left| \frac{\partial^{\alpha+\beta} R_{H,p,r} [f] (x)}{\partial x^\alpha \partial y^\beta} \right| \leq \sum_{j \in I} \sum_{|\kappa| \leq \alpha + \beta} \left( \frac{\alpha}{\kappa_1} \right) \left( \frac{\beta}{\kappa_2} \right) \frac{\partial^{\alpha+\beta-|\kappa|} \tilde{W}_{\mu,j} (x, y) \partial^{|\kappa|} R_{H,p,r}^\Delta : f (x)}{\partial x^{\alpha-\kappa_1} \partial y^{\beta-\kappa_2}} \leq \sum_{|\kappa| \leq \alpha + \beta} \left( \frac{\alpha}{\kappa_1} \right) \left( \frac{\beta}{\kappa_2} \right) C_{\alpha-\kappa_1, \beta-\kappa_2} M \left| f \right|_{p,1} \max_{i \in I} \left( \rho_j^{2+|\kappa|} S_j^{p-|\kappa|} d_j^{p-|\kappa|} \right) \tilde{K}_\kappa (p)
$$

and the result follows by settings (1)-(5).

The following statements can be easily checked.

**Theorem 10** The operator $S_{H,p,r} [\cdot]$ interpolates on all data required for its definition provided that $\mu > r + 1$.

**PROOF.** $H_{p,r}^{\Delta : f } [f]$ interpolates $f$ and all its partial derivatives up to the order $r$ at $x^i$, for each $i = 1, \ldots, N$. Moreover, as pointed out in Sec. 1 the local Shepard basis is cardinal

$$
\tilde{W}_{\mu,i} (x^k) = \delta_{ik}, \ i, k = 1, \ldots, N.
$$

and

$$
\tilde{W}_{\mu,i}^{(\alpha, \beta)} (x^k) = 0, \ k = 1, \ldots, N; (\alpha, \beta) \in \mathbb{Z}_+^2, 1 \leq \alpha + \beta \leq \mu - 1.
$$

**Theorem 11** The algebraic precision of the operator $S_{H,p,r} [\cdot]$ is $p$.

**PROOF.** The algebraic precision of the operator $H_{p,r}^{\Delta : f } [f] (x)$ is $p$ for $i = 1, \ldots, N$ and as pointed out in Sec. 1 the local Shepard basis is a partition of unity.
4 Numerical results

To test the accuracy of approximation of the bivariate Shepard-Hermite operators in the multivariate interpolation of large sets of scattered data, we test the local Shepard operator combined with Hermite interpolation polynomials (27), (29) and (30) on a set of well-known test functions on the unit square $\Omega = [0, 1] \times [0, 1]$.

**Exponential:**

$$ F_1 = 0.75 \exp\left(\frac{-(9x-2)^2+(9y-2)^2}{4}\right) + 0.50 \exp\left(\frac{-(9x-7)^2+(9y-3)^2}{4}\right) + 0.75 \exp\left(\frac{-(9x+1)^2}{49} - \frac{(9y+1)^2}{10}\right) - 0.20 \exp\left(-(9x - 4)^2 - (9y - 7)^2\right), $$

**Cliff:**

$$ F_2 = \frac{\tanh(9y - 9x) + 1}{9}, $$

**Saddle:**

$$ F_3 = \frac{1.25 + \cos(5.4y)}{6 + 6(3x - 1)^2}, $$

**Gentle:**

$$ F_4 = \frac{\exp\left(\frac{-81}{16} ((x - 0.5)^2 + (y - 0.5)^2)\right)}{3}, $$

**Steep:**

$$ F_5 = \frac{\exp\left(\frac{-81}{4} ((x - 0.5)^2 + (y - 0.5)^2)\right)}{3}, $$

**Sphere:**

$$ F_6 = \frac{\sqrt{64 - 81((x - 0.5)^2 + (y - 0.5)^2)}}{9} - 0.5, $$

$$ F_7 = 2 \cos(10x) \sin(10y) + \sin(10xy), $$

$$ F_8 = \exp\left(-\frac{(5-10x)^2}{2}\right) + 0.75 \exp\left(-\frac{(5-10y)^2}{2}\right) + 0.75 \exp\left(-\frac{(5-10x)^2}{2}\right) \exp\left(-\frac{(5-10y)^2}{2}\right). $$
\[ F_9 = \left( \frac{20}{3} \right)^3 \exp\left( \frac{10-20x}{3} \right) \exp\left( \frac{10-20y}{3} \right) \right)^2 \times \left( \frac{1}{1+\exp\left( \frac{10-20x}{3} \right)} \right) \left( \frac{1}{1+\exp\left( \frac{10-20y}{3} \right)} \right)^5 \times \exp\left( \frac{10-20x}{3} \right) - \frac{2}{1+\exp\left( \frac{10-20x}{3} \right)} \times \exp\left( \frac{10-20x}{3} \right) - \frac{2}{1+\exp\left( \frac{10-20y}{3} \right)} , \]

\[ F_{10} = \exp\left( -0.04\sqrt{(80x - 40)^2 + (90y - 45)^2} \right) \times \cos\left( 0.15\sqrt{(80x - 40)^2 + (90y - 45)^2} \right) . \]

For each function of the set we compare numerical results obtained by applying the Shepard-Hermite operators \( S_{H_{p,r}}[f] \) having degree of exactness 2, 3 and 4 with those obtained by applying the local Shepard-Taylor operators \( S_{T_{p}}[f] \) (10) having the same degree of exactness of \( S_{H_{p,r}}[f] \) respectively. In the following we set \( p = \mu = 2, 3, 4 \) and \( r = 1, 2 \). We emphasize that operators \( S_{T_{p}}[f], p = 2, 3, 4 \) make use, at each sample point, of functional and partial derivatives data up to the orders 2, 3 and 4 respectively, while operators \( S_{H_{p,r}}[f] \) make use, at each sample point, of functional evaluation and first order derivatives data for \( p = 2 \), and functional evaluation and first and second order derivatives data for \( p = 3, 4 \). We compute numerical approximations using the Franke dataset [9] and a \( 33 \times 33 \) sparse set of uniformly distributed interpolation nodes in the unit square \( \Omega \). We compute the resulting approximations at the points of a regular grid of \( 101 \times 101 \) points of \( \Omega \). We show in table 1-10 maximum, mean and mean square interpolation errors, computed for the parameter value \( N_w = 30 \).

5 Conclusion

In the context of the problem of the enhancement, using supplementary derivative data, of the algebraic precision of linear operators for univariate and multivariate approximation, in this paper we have defined the local Shepard-Hermite operators. These operators are realized as combination of Shepard operators with Hermite interpolation polynomials on the simplex by a recently established procedure. The local Shepard-Hermite operators \( S_{H_{p,r}}[f] \) allow us to raise the algebraic precision of the Shepard-Taylor operators \( S_{T_{r}}[f] \) that use the same data, maintaining at the same time their interpolation proper-
Table 1
Exponential

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<th>Dataset</th>
<th>Method</th>
<th>Max Error</th>
<th>Mean Error</th>
<th>Mean Square Error</th>
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<td>Franke</td>
<td>$S_{T_2} [f]$</td>
<td>1.2818e - 001</td>
<td>9.3246e - 003</td>
<td>3.2715e - 004</td>
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<td>9.0526e - 003</td>
<td>2.8658e - 004</td>
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<td>$S_{T_3} [f]$</td>
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<td>1.3135e - 003</td>
<td>1.6680e - 005</td>
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<td>1.7363e - 003</td>
<td>2.0254e - 005</td>
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<td></td>
<td>$S_{T_4} [f]$</td>
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<td>4.5481e - 004</td>
<td>1.9222e - 006</td>
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<td></td>
<td>$S_{H_{4,2}} [f]$</td>
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<td>1.4449e - 003</td>
<td>3.2705e - 005</td>
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<td>$33 \times 33$</td>
<td>$S_{T_2} [f]$</td>
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<td>1.1604e - 003</td>
<td>5.2192e - 008</td>
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<tr>
<td></td>
<td>$S_{H_{2,1}} [f]$</td>
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<td>1.5696e - 004</td>
<td>9.7262e - 008</td>
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<tr>
<td></td>
<td>$S_{T_3} [f]$</td>
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**Steep**

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Table 8

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<tbody>
<tr>
<td>Franke</td>
<td>$S_{T_2}$ [f]</td>
<td>0.535700</td>
<td>3.1764e – 002</td>
<td>3.6768e – 003</td>
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<tr>
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<td>$S_{H_{2,1}}$ [f]</td>
<td>0.501500</td>
<td>3.1111e – 002</td>
<td>3.6272e – 003</td>
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<tr>
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<td>$S_{T_3}$ [f]</td>
<td>0.244940</td>
<td>3.8344e – 003</td>
<td>2.4872e – 004</td>
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<td>0.311220</td>
<td>8.1462e – 003</td>
<td>5.5225e – 004</td>
</tr>
<tr>
<td>$33 \times 33$</td>
<td>$S_{T_2}$ [f]</td>
<td>0.034519</td>
<td>1.1204e – 003</td>
<td>7.848e – 006</td>
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<tr>
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<td>$S_{H_{2,1}}$ [f]</td>
<td>0.064591</td>
<td>3.2346e – 003</td>
<td>6.242e – 005</td>
</tr>
<tr>
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<td>$S_{T_3}$ [f]</td>
<td>0.036808</td>
<td>5.1101e – 005</td>
<td>2.5394e – 007</td>
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<tr>
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<td>$S_{H_{3,2}}$ [f]</td>
<td>0.031103</td>
<td>5.9131e – 005</td>
<td>2.3588e – 007</td>
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<tr>
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<td>$S_{T_4}$ [f]</td>
<td>0.040466</td>
<td>1.2254e – 005</td>
<td>1.8864e – 007</td>
</tr>
<tr>
<td></td>
<td>$S_{H_{4,2}}$ [f]</td>
<td>0.022272</td>
<td>2.1962e – 005</td>
<td>1.1813e – 007</td>
</tr>
</tbody>
</table>
ties, and also to achieve the accuracy of approximation of the Shepard-Taylor operators $S_{T_p} [f]$ with the same algebraic precision.

References


[9] www.mathstat.strath.ac.uk/oleg.davydov/Franke_test_data/sddata/ds3.m


