A unified matrix approach
to polynomial sequences:
orthogonal and Sheffer polynomials cases

F. A. Costabile, E. Longo

Department of Mathematics and Computer Science, University of Calabria, Italy
francesco.costabile@unical.it, longo@mat.unical.it

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We propose a unified matrix approach to polynomial sequences (p.s.) \( \{p_n(x)\}_{n \in \mathbb{N}} \), i.e. \( p_n(x) \in \mathcal{P} \ \forall n \in \mathbb{N} \), degree \( p_n(x) = n \), and \( p_0(x) = c \neq 0 \). As example we consider orthogonal and Sheffer p. s.

Let \( \{p_n(x)\}_{n \in \mathbb{N}} \) be a p.s. and

\[
p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k, \quad a_{n,n} \neq 0 \ \forall n \in \mathbb{N},
\]

its decomposition on canonical base. Let \( A \) be the infinite, lower triangular, non singular matrix with entries \( a_{n,k} \), \( n = 0, 1, ..., k = 0, ..., n \), \( n \in \mathbb{R} \) (or \( \mathbb{C} \)) and \( B = A^{-1} \) with entries \( b_{n,k} \), \( n = 0, 1, ..., k = 0, ..., n \). Let \( X(x), P(x) \) be the infinite vectors, respectively,

\[
X(x) = [1, x, ..., x^n]^T, \quad P(x) = [p_0(x), p_1(x), ..., p_n(x), ...]^T.
\]

Then we have

\[
P(x) = AX(x), \quad (3)
\]
\[
X(x) = BP(x). \quad (4)
\]

From (4) we have

\[
x^n = \sum_{k=0}^{n} b_{n,k} p_k(x), \quad \forall n \in \mathbb{N}. \quad (5)
\]

Corollary 1. The p. s. \( \{p_n(x)\}_{n \in \mathbb{N}} \) is a base of \( \mathcal{P}_n \).

Theorem 1. A set of polynomials \( p_0(x), p_1(x), ..., p_n(x), ... \) is a p. s. \( \{p_n(x)\}_{n \in \mathbb{N}} \)
if there exist entries \( (b_{n,k})_{n=0,1,..., k=0,...,n} \) with \( b_{n,n} \neq 0 \ \forall n \in \mathbb{N} \) such that

\[
p_0(x) = \frac{1}{b_{0,0}}, \quad (6)
\]

\[
p_n(x) = \frac{(-1)^n}{\prod_{i=0}^{n} b_{i,i}} \begin{vmatrix}
1 & x & \cdots & x^{n-1} & x^n \\
0 & b_{1,0} & \cdots & b_{n-1,0} & b_{n,0} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & b_{n-1,n-1} & b_{n,n}
\end{vmatrix}, \quad (7)
\]
\[ p_n(x) = \frac{1}{b_{n,n}} \left( x^n - \sum_{k=0}^{n-1} b_{n,k} p_k(x) \right), \quad n = 1, 2, \ldots. \]  

**Theorem 2.** Any p. s. \( \{\hat{p}_n(x)\}_{n \in \mathbb{N}} \) is solution of linear interpolation problem

\[ L_i(p_n) = \delta_{n,i}, \quad i = 0, \ldots, n, \]  

where \( L_i, i = 0, \ldots, n, \) are linearly independent linear functionals on \( \mathcal{P}. \)

Let \( Q = (q_{i,j}) \) and \( \hat{Q} = (\hat{q}_{i,j}) \) be the production matrices ( [5]) for \( A \) and \( B, \) respectively, i. e.

\[ AQ = DQ, \quad B\hat{Q} = D\hat{Q}, \quad \text{with} \quad D = (\delta_{i+1,j}), \quad i,j = 0, \ldots, n. \]  

The we have

**Theorem 3.** For p. s. \( \{p_n(x)\}_{n \in \mathbb{N}} \) the following relations hold:

\[ \hat{q}_{n,n+1} p_{n+1}(x) = - (\hat{q}_{n,n} - x) p_n(x) - \hat{q}_{n,n-1} p_{n-1}(x) - \cdots - \hat{q}_{n,0} p_0(x), \]  

\[ p_{n+1}(x) = \frac{(-1)^{n+1}}{\prod_{i=0}^{n} \hat{q}_{i,i+1}} \begin{vmatrix} \hat{q}_{0,0} - x & \hat{q}_{0,1} & 0 & \cdots & 0 \\ \hat{q}_{1,0} & \hat{q}_{1,1} - x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{q}_{n,0} & \cdots & \cdots & \hat{q}_{n,n} - x \end{vmatrix}. \]  

In the following we consider the application of this theory to orthogonal and Sheffer polynomials cases.

**References**


