A new determinant form of Bessel polynomials and application

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Keywords. function, polynomial sequence, Bessel, determinant form

The Bessel polynomials are the unique polynomial solutions, with constant equal to 1, of the differential equation

\[ x^2 y''(x) + (2x + 2)y'(x) - n(n + 1)y(x) = 0 \] (1)

They are important in certain problems of mathematical physics, for example they arise in the study of electrical networks and when the wave equation is considered on spherical coordinates. Many application may be found in [7]. In recent years new interests are arisen in determinant form of Bessel polynomials [9-11]. In this paper we propose a new determinant form with related recurrence relation. Some application are considered, too.

The Bessel polynomials are given in [1] in the form

\[
\begin{align*}
y_0(x) &= 1 \\
y_n(x) &= \sum_{k=0}^{n} \frac{(n+k)!}{2^k(n-k)!k!} x^k
\end{align*}
\] (2)

Putting

\[ b_{n,k} = \frac{(n+k)!}{2^k(n-k)!k!} \quad n = 0, 1, \ldots, \quad 0 \leq k \leq n \] (3)

we have the infinite lower triangular matrix \( B = (b_{n,k}) \).

Then if \( Y(x) = [y_0(x), \ldots, y_n(x), \ldots]^T \), \( X(x) = [1, x, \ldots, x^n, \ldots]^T \) we have

\[ Y(x) = BX(x). \]

Theorem 1. The matrix \( B \) is non singular and putting \( B^{-1} = (b_{n,k}) \) it is

\[
\begin{align*}
b_{n,n}^{-1} &= \frac{2^n n!}{(2n)!} \\
b_{i,j}^{-1} &= -\sum_{k=j}^{i-1} \binom{i}{k} 2^{i-k}(i+k-j)2^{k-j}b_{k,j}^{-1}
\end{align*}
\] (4)

Theorem 2. For Bessel polynomials the following recurrence relation holds

\[ y_n(x) = \frac{(2n)!}{2^n n!} \left( x^n - \sum_{k=0}^{n-1} b_{n,k}^{-1} y_k(x) \right) \] (5)
Corollary 1. The Polynomials sequence \( \{y_n(x)\} \) is a bases for \( P_n \).

Theorem 3. Let \( \{y_n(x)\} \) the Bessel polynomial sequence as defined in (5), then the following identity holds

\[
y_n(x) = \frac{(-1)^n}{\prod_{i=0}^{n} b_{i,i}} \begin{vmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ b_{0,0} & b_{1,0} & \cdots & b_{n-1,0} & b_{n,0} \\ b_{1,1} & b_{2,1} & \cdots & b_{n-1,1} & b_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,n-1} & b_{n,n-1} \end{vmatrix} \tag{6}
\]

Theorem 4. The Bessel polynomial \( y_n(x) \) is the unique solution of the linear interpolation problems

\[
L_i(p_n) = \delta_{n,i} \quad i = 0, \ldots, n
\]
with \( L_i(x^k) = b_{k,i}^{-1} \).

Theorem 5. For the Bessel function of order \( n + \frac{1}{2} \) we have the determinant form

\[
J_{n+\frac{1}{2}}(r) = (2\pi r)^{-1/2} \frac{(-1)^{n^2+n-1}e^{-ir}}{\prod_{i=0}^{n} b_{i,i}^{-1}} \begin{vmatrix} 1 & \cdots & (-\frac{1}{ir})^n \\ -\frac{1}{ir} & \cdots & b_{n,0}^{-1} \\ \vdots & \ddots & \vdots \\ b_{n-1,n-1}^{-1} & \cdots & b_{n,n-1}^{-1} \end{vmatrix} + \frac{(-1)^{n^2+n+1}e^{-ir}}{\prod_{i=0}^{n} b_{i,i}^{-1}} \begin{vmatrix} 1 & \cdots & (\frac{1}{ir})^n \\ \frac{1}{ir} & \cdots & b_{n,0}^{-1} \\ \vdots & \ddots & \vdots \\ b_{n-1,n-1}^{-1} & \cdots & b_{n,n-1}^{-1} \end{vmatrix} \tag{7}
\]

We define the polynomials sequence

\[
\hat{y}_n(x) = \sum_{k=0}^{n} b_{n,k}^{-1}x^k \tag{8}
\]
then we have

Theorem 6. With umbral composition we have

\[
\hat{y}_n(y_n(x)) = y_n(\hat{y}_n(x)) = x^n \tag{9}
\]

After (8) we see that the sequences \( y_n(x) \) and \( \hat{y}_n(x) \) are conjugate polynomial sequence.
References