Spectral analysis and structure preserving preconditioners for fractional diffusion equations

M. Donatelli\textsuperscript{a}, M. Mazza\textsuperscript{a}, S. Serra-Capizzano\textsuperscript{a,b}

\textsuperscript{a}Department of Science and High Technology, University of Insubria, Como, Italy
\textsuperscript{b}Department of Information Technology, division of Scientific Computing, Uppsala University, Uppsala, Sweden
marco.donatelli@uninsubria.it mariarosa.mazza@uninsubria.it stefano.serrac@uninsubria.it

Keywords. fractional diffusion equations, Toeplitz matrix, locally Toeplitz sequence of matrices, singular value/eigenvalue distribution, preconditioning

We are interested in the following initial-boundary value problem of fractional partial diffusion equations (FDEs)

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = d_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + d_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^{-\alpha}} + f(x,t), \ (x,t) \in (L,R) \times (0,T], \\
u(L,t) = u(R,t) = 0, \quad t \in [0,T], \\
u(x,0) = u_0(x), \quad x \in [L,R],
\end{cases}
\]

where \( \alpha \in (1,2) \) is the fractional derivative order, \( f(x,t) \) is the source term, \( \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \) are the right-handed (+) and the left-handed (-) fractional derivatives and the nonnegative functions \( d_{\pm}(x,t) \) are the diffusion coefficients. The FDEs emerges in many topics such as image processing and turbulent flow and are used to describe diffusion phenomena that cannot be modeled by the second order diffusion equations. Fixed two positive integers \( N, M \), and defined the following partition of \([L,R] \times [0,T]\), i.e.,

\[x_i = L + i\Delta t, \quad \Delta x = \frac{(R-L)}{N+1}, \quad i = 0, \ldots, N+1,
\]

\[t_m = m\Delta t, \quad \Delta t = \frac{T}{M}, \quad m = 0, \ldots, M,
\]
in [2,3] it has been introduced an unconditionally stable method for approximating the FDE problem (1) which leads to the following Toeplitz-like system (see [6]):

\[
\begin{pmatrix}
\nu_{M,N} I + D_+^{(m)} T_{\alpha,N} + D_-^{(m)} T_{\alpha,N}^T
\end{pmatrix} u^{(m)} = \nu_{M,N} u^{(m-1)} + \Delta x^\alpha f^{(m)},
\]

where \( u^{(m)} = [u_1^{(m)}, \ldots, u_N^{(m)}]^T \) with \( u_i^{(m)} \) a numerical approximation of \( u(x_i,t_m) \),

\[D_\pm^{(m)} = \text{diag}(d_{\pm,1}^{(m)}, \ldots, d_{\pm,N}^{(m)})\] with \( d_{\pm,i}^{(m)} := d_{\pm}(x_i,t_m), \ f^{(m)} = [f_1^{(m)}, \ldots, f_N^{(m)}]^T\]
with \( f_i^{(m)} := f(x_i, t_m) \), \( v_{M,N} = \frac{\Delta x^N}{2^N} \), \( I \) is the identity matrix of order \( N \) and

\[
T_{\alpha,N} = \begin{bmatrix}
g^{(\alpha)}_1 & g^{(\alpha)}_0 & 0 & \cdots & 0 & 0 \\
g^{(\alpha)}_2 & g^{(\alpha)}_1 & g^{(\alpha)}_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
g^{(\alpha)}_{N-1} & \cdots & \cdots & g^{(\alpha)}_1 & g^{(\alpha)}_0 & 0 \\
g^{(\alpha)}_N & g^{(\alpha)}_{N-1} & \cdots & \cdots & g^{(\alpha)}_2 & g^{(\alpha)}_1 \\
\end{bmatrix}_{N \times N}
\]

is a lower Hessenberg Toeplitz matrix whose coefficients \( g^{(\alpha)}_k \) are defined as

\[
g^{(\alpha)}_k = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k}{k!} \alpha(\alpha-1) \cdots (\alpha-k+1) k = 0,1,\ldots
\]

with the formal notation \( \binom{\alpha}{0} = 1 \).

In the past three years, preconditioners and multigrid methods have been proposed for solving the linear system (1) when \( d_+(x,t) = d_1 > 0 \) and \( d_-(x,t) = d_2 > 0 \), i.e., in the case of Toeplitz matrices. For example, in [1] the authors used the CGNR method with a circulant preconditioner and verified that it converges superlinearly, while in [4] a V-grid method is employed and the linear convergence of the corresponding two-grid method is proved in the case of constant and equal diffusion coefficients, i.e. \( d_\pm(x,t) = d > 0 \).

In this work we show that in the case of nonconstant diffusion coefficients, the resulting matrix-sequence is a generalized locally Toeplitz (GLT) (see [5]) and then we use the GLT machinery to study its singular value/eigenvalue distribution as the matrix size diverges.

On the one hand, our new spectral information are employed to show that the superlinear convergence obtained in the constant coefficient case for the CGNR method with a circulant preconditioner cannot be replicated by any Krylov method in the nonconstant coefficient case. Furthermore, we observe that the circulant preconditioning strategy cannot ensure a superlinear convergence for a Krylov method even in the multidimensional setting.

On the other hand, we exploit our spectral tools to extend the convergence analysis provided in [4] showing that its linear character is preserved also in the nonconstant coefficient case. Moreover, we theoretically justify the linear convergence experimentally observed for the V-cycle method. Thanks to known results on multigrid methods for multilevel matrix algebras, we expect the same behaviour also in the multidimensional case.

Finally, we propose two structure preserving preconditioners with minimal bandwidth (and so with efficient computational cost) in combination with CGNR and GMRES methods. The first preconditioner is defined as

\[
P^{(m)}_{1,N} = v_{M,N} I + D_+^{(m)} B_N + D_-^{(m)} B_N^T,
\]
where $B_N$ is an approximation of the first derivative operator, while the second preconditioner is given by

$$D^{(m)}_{2,N} = \nu_{M,N} I + D^+_N L_N + D^-_N L^T_N,$$

where $L_N$ is the Laplacian matrix. We give numerical evidence that $D^{(m)}_{1,N}$ is suitable for fractional exponents close to 1, while $D^{(m)}_{2,N}$ is recommended for fractional exponents close to 2, and in general that both are more effective than the recently used circulant preconditioner.

References