Hyperholomorphic Special Functions, their relations to multivariate Appell sequences and combinatorics

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1 Motivation

In the beginning of the 1930s, Rudolf Fueter, renowned number theorists and disciple of David Hilbert, initiated the systematical creation of Quaternionic Analysis as a hypercomplex generalization of complex function theory (cf. [6]). The application in Mathematical Physics were not in the center of his attention. However, the use of general Clifford algebras in multivariate analysis only started to grow significantly in the 70s of the last century.

The systematic treatment of hyperholomorphic functions (solutions of a system of generalized Cauchy-Riemann equations) by their multiple series expansion, based on Gegenbauer or associated Legendre polynomials called immediately attention to the role of Special Functions in this context. But also algebraic approaches to Special Functions by means of Lie algebras or through Lie groups and symmetric spaces influenced their consideration in Clifford Analysis [2].

Here we apply the classical approach to Special Functions through differential equations, but now with respect to the hypercomplex derivative ( [7], see also [3]), which is a more recently developed tool in Clifford Analysis. In this way we can stress the role of hypercomplex Special Functions as intermediator between continuous and discrete mathematics. The hypercomplex exponential function leads to generalized Appell polynomial sequences (see [1,5]). Since many algebraic structures have hidden combinatorial underpinnings, we show at the end some of their relations to combinatorial topics.

2 Basic notations

We use an orthonormal base \( \{e_1, e_2, \cdots, e_n\} \) of the Euclidean vector space \( \mathbb{R}^n \) equipped with a product according to the multiplication rules \( e_k e_l + e_l e_k = -2\delta_{kl}, \ k, l = 1, \cdots, n \), where \( \delta_{kl} \) is the Kronecker symbol. This non-commutative product generates the \( 2^n \)-dimensional Clifford algebra \( Cl_{0,n} \) over \( \mathbb{R} \) and the set
\{e_A : A \subseteq \{1, \cdots, n\}\} with \(e_A = e_{h_1}e_{h_2} \cdots e_{h_r},\) \(1 \leq h_1 \leq \cdots \leq h_r \leq n,\) \(e_\emptyset = e_0 = 1,\) forms a base of \(\text{Cl}_{0,n}.\) The real vector space \(\mathbb{R}^{n+1}\) will be embedded in \(\text{Cl}_{0,n}\) by identifying the element \((x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}\) with the para-vector \(x = x_0 + x,\) where \(x = e_1x_1 + \cdots + e_nx_n.\) Like in the complex case, the conjugate of \(x\) is \(\bar{x} = x_0 - x\) and the norm \(|x|\) of \(x\) is defined by \(|x|^2 = x\bar{x} = x_0^2 + x_1^2 + \cdots + x_n^2.\) We also use \(\omega(x) := \frac{x}{|x|^2}.\)

3 Hyperholomorphic functions

We consider \(\text{Cl}_{0,n}\)-valued functions in some open subset \(\Omega \subset \mathbb{R}^{n+1},\) of the form \(f(z) = \sum_A f_A(z)e_A,\) where \(f_A(z)\) are real valued. We suppose that \(f\) is hypercomplex differentiable in \(\Omega\) in the sense of [7], i.e. has a uniquely defined areolar derivative \(f'\) in each point of \(\Omega\) (for the definition of an areolar derivative see [3]). Then \(f'\) can be expressed by \(f' = 1/2(\partial_0 - \partial_x)f,\) where \(\partial_0 := \frac{\partial}{\partial x_0}, \partial_x := e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial x_n}.\) If now \(D := \partial_0 + \partial_x\) is the usual generalized Cauchy-Riemann differential operator (cf. [2]), then, obviously \(f' = 1/2 \bar{D}f.\) Since in [7] it has been shown that a hypercomplex differentiable function satisfies the equation \(Df = 0 (f\) is a hyperholomorphic function in the sense of Clifford Analysis), then \(f' = \partial_0 f\) like in the complex case.

4 Outlook

The following results (cf. [3] and [8]) illustrate the objects of our talk.

(i) The hyperholomorphic solution \(E = E(x),\) of the initial value problem

\[
\begin{align*}
E' &= E \quad (1) \\
E(0) &= 1. \quad (2)
\end{align*}
\]

in \(\mathbb{R}^{n+1},\) is given by the hypercomplex exponential function

\[
E(x) = \text{Exp}_n(x_0 + x) = e^{x_0}I_n \left( \frac{2}{|x|} \right) ^{\frac{|x|}{2}} \left( J_{\frac{|x|}{2}}(\omega(x)) + \omega(x)J_{\frac{|x|}{2}}(\omega(x)) \right).
\]

(ii) The hypercomplex exponential function generates through

\[
E(x) = \text{Exp}_n(xt) = \sum_{k=0}^{\infty} \frac{P_n^k(x) t^k}{k!}
\]

the generalized (with respect to the hypercomplex derivative) Appell sequence \((P_n^k(x))_{k \geq 0}\) with \(P_0^k(x) \equiv 1.\)

(iii) The explicit expression of \(P_n^k(x)\) is given by

\[
P_n^k(x) = \sum_{s=0}^{k \atop s} \binom{k}{s} c_s(n) x_0^{k-s} x_s.
\]
where \( c_k(n) := \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!}, & \text{if } k \text{ is odd} \\ c_{k-1}(n), & \text{if } k \text{ is even} \end{cases} \) (3)

(iv) For the simplest non-complex case \( n = 2 \) the coefficients \( c_k(2) \) are related to the generalized central binomial coefficients in the form

\[
c_k(2) = \frac{1}{2^k} \left( \frac{k}{\lfloor k/2 \rfloor} \right), \quad \text{where } \lfloor \rfloor \text{ is the floor function}. \quad (4)
\]

The talk will be dedicated to numerical and combinatorial properties of the generalized Appell sequences, particularly to new combinatorial identities, thereby continuing our investigations published in [4]).

References