Initial-boundary value problems for diffusion equations and approximation by positive linear operators

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Very recently a comprehensive series of results (a theory) has been developed and documented in the monograph [1]. Among other things, this book is concerned with several classes of initial-boundary value problems for diffusion equations on a convex compact subset $K$ of $\mathbb{R}^d$, $d \geq 1$, having non-empty interior, the main aim being that one of approximating their solutions by means of positive linear operators and of studying their regularity (spatial) properties.

The lecture will report some of the main results. In particular, it will focus on a sequence of positive linear operators on $C(K)$ which are referred to as Bernstein-Schnabl operators. These operators, which generalize the classical Bernstein operators, not only furnish new general approximation processes for continuous functions on $K$ but they also approximate the Markov semigroups which govern initial-boundary value problems for (degenerate) diffusion equations of the form

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = A_T(u(\cdot,t))(x), \ (x \in K, \ t > 0) \\
u(x,0) = u_0(x), \quad u_0 \in D(A_T),
\end{cases}
$$

(1)

where $A_T$ is the closure of the differential operator $(W_T, C^2(K))$ defined as

$$
W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j},
$$

$T : C(K) \to C(K)$ is a given Markov operator and

$$
\alpha_{ij} := T(pr_ipr_j) - (pr_ipr_j) \ (i, j = 1, \ldots, d).
$$

The boundary conditions for problem (1) are incorporated in the domain $D(A_T)$. They include the so-called Wentzel’s boundary conditions

$$
A_T u = 0 \quad \text{on } \partial_T K \quad (u \in D(A_T)).
$$

(2)
Initial-boundary value evolution problems of the form (1) occur in the study of diffusion problems arising from different areas such as biology, mathematical finance and physics.

Both approximation and shape preserving properties of Bernstein-Schnabl operators will be discussed together with their counterparts for the relevant Markov semigroups which, in turn, lead to spatial regularity properties to the solutions $u(\cdot, t), t \geq 0$, of problems (1). In particular, the asymptotic behaviour

$$\lim_{t \to +\infty} u(x, t) \quad \text{uniformly with respect to } x \in K$$

is determined as well.

**References**